Third Quiz for CSI35

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Directions: This quiz is due Monday March 16, at 4:00 PM.

1. Let $A = \{0, 1\}$.
   (a) How many (binary) relations are there on A? List all of them.
   (b) Which of the relations you listed in par (a) are reflexive? Which are symmetric? Which are antisymmetric? Which are transitive?

2. The relation R on the set of real numbers $\mathbb{R}$ is defined as follows:

$$R = \{(x, y) \in \mathbb{R}^2 : x^4 = y^4\}$$

   Prove that:
   (a) R is reflexive.
   (b) R is symmetric.
   (c) R is transitive.

3. Consider the following relation on the set of natural numbers $\mathbb{N}$:

$$R = \{(m, n) : m \text{ divides } n\}$$

   (a) R is reflexive.
   (b) R is antisymmetric.
   (c) R is transitive.

4. Let R_1 and R_2 be the relations on $\{1, 2, 3\}$ represented the digraphs G_1 and G_2 shown in Figure 1.
   (a) Find the matrices M_1 and M_2 representing the relations R_1 and R_2.
   (b) Write down the relations R_1 and R_2 as sets of ordered pairs.
   (c) Find the matrices corresponding to the relations $R_1 \circ R_2$ and $R_2 \circ R_1$.

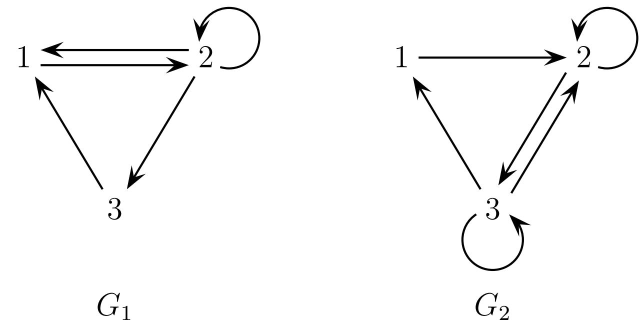

Figure 1: The digraphs of Question 4

(d) Write the relations $R_1 \circ R_2$ and $R_2 \circ R_1$ as sets of ordered pairs.

(e) Draw the digraphs representing the relations $R_1 \circ R_2$ and $R_2 \circ R_1$.

5. Consider the following zero-one matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Prove that $A^n = A$ for all natural numbers $n \geq 1$, where the power is with respect to the boolean product.

6. **Extra Credit** After Alice and Bob finished playing a game they decided to study for the Discrete Mathematics exam that was coming up. After a while they had the following conversation:

A: I was looking at the definition of an equivalence relation and it seems redundant.

B: How so?

A: Well, it says that an equivalence relation is a relation that is reflexive, symmetric and transitive. Now that's redundant, because I can prove that if a relation is symmetric and transitive then it is reflexive as well. So to be economical we should define an equivalence relation to be a relation that is symmetric and transitive. No need to check for reflexivity, really.

B: Hm, this sound fishy to me. Somebody would've noticed before. Let me see your proof.

A: It's very simple really: Let R be a symmetric and transitive relation on a set A. To prove that it is reflexive I need to prove that for all $x \in A$ we have $(x, x) \in R$. So let $x \in R$, chose any $y \in A$ such that $(x, y) \in R$. Then since R is symmetric we have $(y, x) \in R$ also. So we have $(x, y) \in R$ and $(y, x) \in R$, so since R is transitive we conclude that $(x, x) \in R$. So, R is reflexive.

Bob is thoughtful for a while.

B: Hm, your proof seems valid... Still, I find it hard to believe that nobody had noticed this before. Let me think some more.

Bob thinks some more.

B: Ok, your proof must be wrong because I can produce a counter example. Remember the matrix in exercise 5:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

This matrix is symmetric and transitive, but not reflexive.

A: (after some thought) Yep, you're right. Still though I can't figure out where is the gap in my proof.

B: Me neither ...

(a) Prove that Bob's counter-example really works.

(b) Can you find the fault in Alice's proof?