Midterm exam

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Directions: Please write your answers in separate papers and staple all the papers together. This exam is due Monday, March 23, at 4:00 PM.

1. Prove by induction that for all natural numbers $n \geq 1$ the following identity holds:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

2. In the Land of Oz, they have only 3-cent and 8-cent stamps. Prove that you can use combinations of these stamps to pay for any letter that costs 14 or more cents.

3. Recall the recursive definition of the Fibonacci numbers f_n:

$$f_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f_{n-1} + f_{n-2} & \text{if } n \geq 2 \end{cases}$$

   Prove that for all natural numbers $n \geq 1$ we have:

$$f_{n+3} = 3f_n + 2f_{n-1}$$

4. The Fina Bocci company breeds worms for fishing. After each worm is two weeks old they cut off its tail which becomes a new worm. The tail grows back in a week, so once a worm becomes two weeks old it produces a new worm every week. Assuming that no worm ever dies and that the company starts with one newly "born" worm, find a recursive formula for the the number of worms after n weeks.

5. Consider the alphabet $\Sigma = \{0, 1, 2\}$. A *palindrome* is a string $s \in \Sigma^*$ with the property that it reads the same when read backwards. For example "210012" is a palindrome. Let Π be the set of all palindromes in Σ^*.
   (a) Give a recursive definition of Π.
   (b) Find a formula that gives the number of elements of Π of length n, for $n \in \mathbb{N}$.
   (c) **Extra Credit:** Prove the formula you gave in part (b).

6. For a rooted tree T let $v(T)$ and $e(T)$ denote the number of vertices and edges of T respectively. Use structural induction to prove that

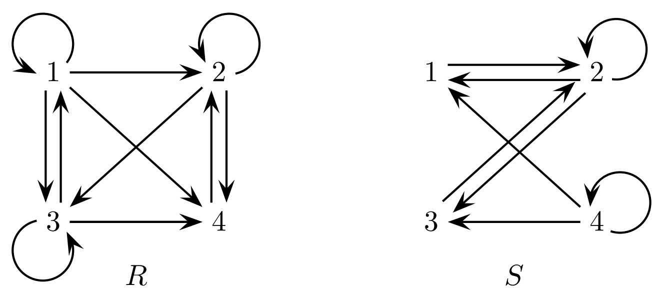
$$v(T) - e(T) = 1$$

7. Consider the relation R represented by the matrix

$$M_R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

   (a) Is R reflexive?
   (b) Is R symmetric?
   (c) Is R transitive?
   (d) Draw the digraph representing R.

8. Consider the relations R, and S on the set $\{1, 2, 3, 4\}$ represented by the digraphs:



   (a) Find the matrices M_S and M_R.
   (b) Use these matrices to compute the compositions $R \circ S$ and $S \circ R$.
   (c) Draw the digraphs that represent $R \circ S$ and $S \circ R$.
   (d) Write the relations $R \circ S$ and $S \circ R$ as sets of ordered pairs.

9. Let $\mathbb{R}^+$ be the set of positive real numbers. Define a relation R on $\mathbb{R}^+$ as follows:

$$R = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : \frac{y}{x} \in \mathbb{Q}\}$$

   where $\mathbb{Q}$ is the set of all rational numbers.

   (a) Prove that R is an equivalence relation.
   (b) Find $[1]$ and $[\sqrt{2}]$, i.e. the equivalence class of 1 and the equivalence class of $\sqrt{2}$.

10. How many equivalence relations are there on the set $\{1, 2, 3, 4, 5, 6\}$?

    Hint. It's easier to count how many partitions of the set $\{1, 2, 3, 4, 5, 6\}$ are there.

11. **Extra Credit:** Consider a $2 \times n$ checkerboard, and let t_n be the number of ways that we can completely tile the board using dominoes. For example as we see in Figure 1 we have $t_1 = 1$, $t_2 = 2$, and $t_3 = 3$. Find a recursive formula for t_n and prove that it is correct.

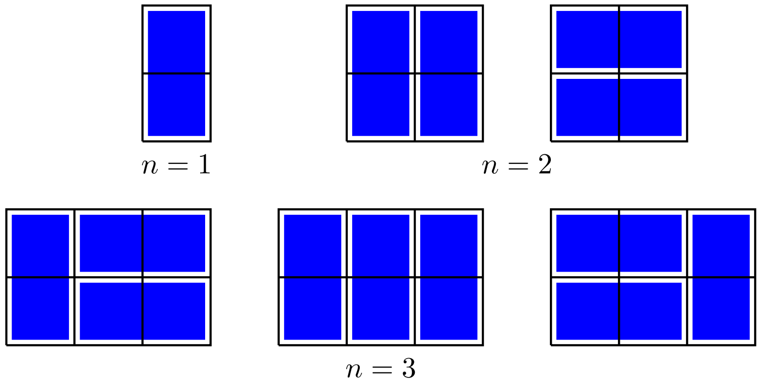


Figure 1: Tilling an $2 \times n$ board with dominoes for $n = 1, 2, 3$