Fifth set of Homework

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Please note: You should fully justify your answers.

Rational root theorem and solving polynomial equations

1. Find a polynomial of degree 3, roots at $x = \pm 3i$ and $x = 2$, and leading coefficient 7.
2. Find a polynomial of degree 5, with integer coefficients, a double root at $x = \dfrac{3}{5}$, and single roots at $x = 1, -1, 3$.
3. List all possible rational roots of the following polynomials according to the "Rational zero theorem".
   (a) $p(x) = x^3 + 3x^2 - 5x - 60$
   (b) $q(x) = 2x^7 - 5x^6 + 2x^2 + 3x - 21$
   (c) $g(x) = 12x^4 - 15x^3 - 4x^2 + x + 6$
   (d) $f(x) = 3x^6 + 5x^5 - 8x^4 + 3x^3 - 2x^2 + 11x - 40$
   (e) $p(x) = 10x^6 - 19x^5 + 6x^4 - 10x^2 + 19x - 6$
4. Solve the following polynomial equations.
   (a) $x^3 + 6x^2 - x - 30 = 0$
   (b) $x^4 + 3x^3 - 16x^2 + 19x - 7 = 0$
   (c) $x^3 + 9x^2 + 27x + 27 = 0$
   (d) $x^4 + x^3 - 7x^2 - x + 6 = 0$
   (e) $x^4 + x^3 - 11x^2 + 9x - 180 = 0$
   (f) $x^5 - x^4 - 5x^3 + x^2 + 8x + 4 = 0$
   (g) $x^4 - 7x^3 + 13x^2 + 3x - 18 = 0$
   (h) $x^8 - 2x^7 - 9x^6 + 12x^5 + 27x^4 - 18x^3 - 31x^2 + 8x + 12 = 0$
   (i) $6x^3 + 41x^2 - 8x - 7 = 0$
   (j) $10x^4 + 29x^3 - 15x^2 - 5x + 2 = 0$
   (k) $12x^4 + 92x^3 + 43x^2 - 88x + 21 = 0$
   (l) $10x^6 - 19x^5 + 6x^4 - 10x^2 + 19x - 6 = 0$
5. **Extra Credit:** Prove using the Rational Root Theorem to prove that $\sqrt[3]{5}$ is irrational.
6. **Extra Credit:** Prove that $3 - \sqrt{2}$ is irrational.
7. **Extra Credit:** Prove that a polynomial of odd degree has at least one real root.
8. **Extra Credit:** Let a, b, c be real numbers. Assume that all the roots of the following polynomial $p(x)$ are rational. Prove $p(x)$ has at least one multiple root.

$$p(x) = x^5 + ax^4 + bx^3 + cx^2 - 2x + 13$$