Third exam

The answers

1. Let $f(x) = x^2 - 3x + 2$ and $g(x) = x - 2$. Find $f \circ g$.

   Solution. We have:

   $$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= (x-2)^2 - 3(x-2) + 2 \\ &= x^2 - 4x + 4 - 3x + 6 + 2 \\ &= x^2 - 7x + 12\end{aligned}$$

   □

2. Verify that $f(x) = \dfrac{x+2}{5}$ and $g(x) = 5x - 2$ are a pair of inverse functions.

   Solution. We need to verify that

   - For all x, $f(g(x)) = x$, and
   - For all x, $g(f(x)) = x$

   For the first condition we have:

   $$\begin{aligned}f(g(x)) &= \frac{(5x-2)+2}{5} \\ &= \frac{5x}{5} \\ &= x\end{aligned}$$

   and for the second:

   $$\begin{aligned}g(f(x)) &= 5 \cdot \frac{x+2}{5} - 2 \\ &= x + 2 - 2 \\ &= x\end{aligned}$$

   □

3. Let $f(x) = 2^{x+1}$.

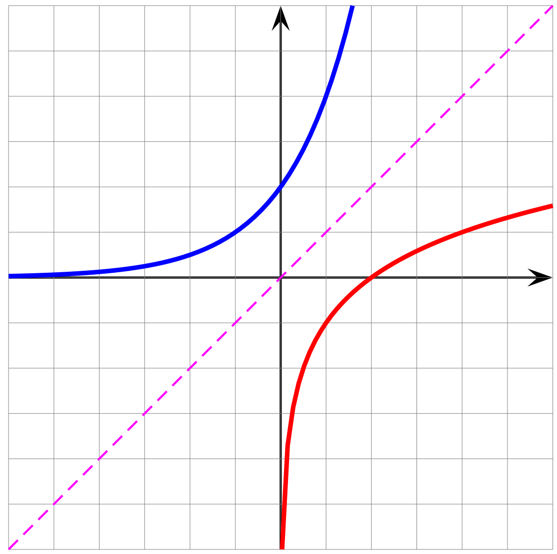
   (a) Find the inverse function f^{-1}.

   Solution. f expressed as a relation is $y = 2^{x+1}$. It's inverse relation is then $x = 2^{y+1}$. To express this relation as a function we have to solve it for y:

   $$\begin{aligned}x = 2^{y+1} &\iff \log_2 x = \log_2 2^{y+1} \\ &\iff \log_2 x = y + 1 \\ &\iff \log_2 x - 1 = y\end{aligned}$$

   So the inverse function of f is given by $f^{-1}(x) = \log_2 x - 1$ □

(b) Sketch both $y = f(x)$ and $y = f^{-1}(x)$ on the same set of coordinates.



4. Solve: $x^3 + 3x^2 - 9x + 5 = 0$

Solution. We'll try to find rational solutions. According to the Rational Zero Theorem the only possible rational roots of this equation are the divisors of its constant term 5. So the only possible rational roots are $\{\pm 1, \pm 5\}$.

We first try $x = 1$:

$$\begin{array}{r|rrrr} & 1 & 3 & -9 & 5 \\ 1 & & 1 & 4 & -5 \\ \hline & 1 & 4 & -5 & 0 \end{array}$$

So $x = 1$ is a solution. We try it again with the quotient to see if it is a double solution:

$$\begin{array}{r|rrr} & 1 & 4 & -5 \\ 1 & & 1 & 5 \\ \hline & 1 & 5 & 0 \end{array}$$

So $x = 1$ is a double solution and the quotient of the original polynomial by $(x-1)^2$ is $x + 5$. It follows that the original equation is equivalent to

$$(x-1)^2(x+5) = 0$$

So the solutions are $x = 1$ (double), and $x = -5$. □

Remark. In the last question, after finding the first solution $x = 1$ we could have used the quadratic formula (or factoring) to solve the quotient $x^2 + 4x - 5$.

5. Solve: $\log_3(2x+1) - \log_3(x-2) = 2$

Solution. We first find the domain of the functions involved to see when the equation is defined: all expressions that appear as arguments to logarithms have to be positive, so we need $2x+1>0$ and $x-2>0$. The first condition is equivalent to $x>-\dfrac{1}{2}$ and the second to $x>2$. The equation is therefore defined when $x>2$, that is in the interval $(2,\infty)$.

Now we solve by first contracting the LHS and then exponentiation both sides:

$$\begin{aligned}
\log_3(2x+1)-\log_3(x-2)=2 &\iff \log_3\left(\frac{2x+1}{x-2}\right)=2\\
&\iff 3^{\log_3\left(\frac{2x+1}{x-2}\right)}=3^2\\
&\iff \frac{2x+1}{x-2}=9\\
&\iff 2x+1=9(x-2)\\
&\iff 2x+1=9x-18\\
&\iff 19=7x\\
&\iff \frac{19}{7}=x
\end{aligned}$$

Since $\dfrac{19}{7}>2$ this solution lies in the domain of the equation and is accepted. $\square$

6. Sketch a graph of the rational function:

$$f(x)=\frac{2x}{x^2-1}$$

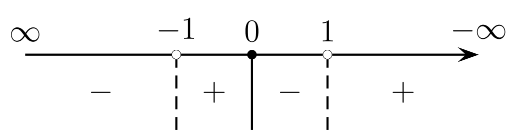
The graph should correctly reflect end behavior, x and y intercepts, and possible asymptotes

Solution. Since the denominator has larger degree than the numerator the x–axis will be a horizontal asymptote.

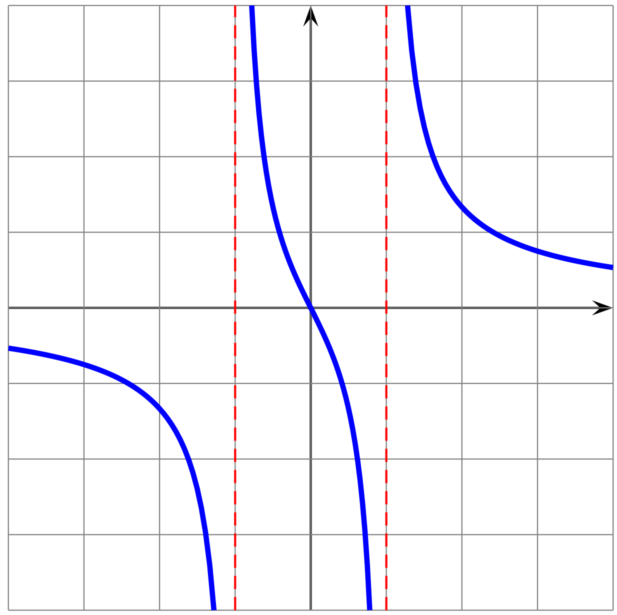
Vertical asymptotes are located at the roots of the denominator, that are not roots of the numerator. The denominator factors as $x^2-1=(x-1)(x+1)$ so we have two vertical asymptotes $x=1$ and $x=-1$.

The y-intercept is $f(0)=0$. The x-intercepts are the roots of the numerator so there is only one x-intercept $x=0$ (which happens to also be the y-intercept).

We have the following table of signs:



Putting all this together we get the following graph:

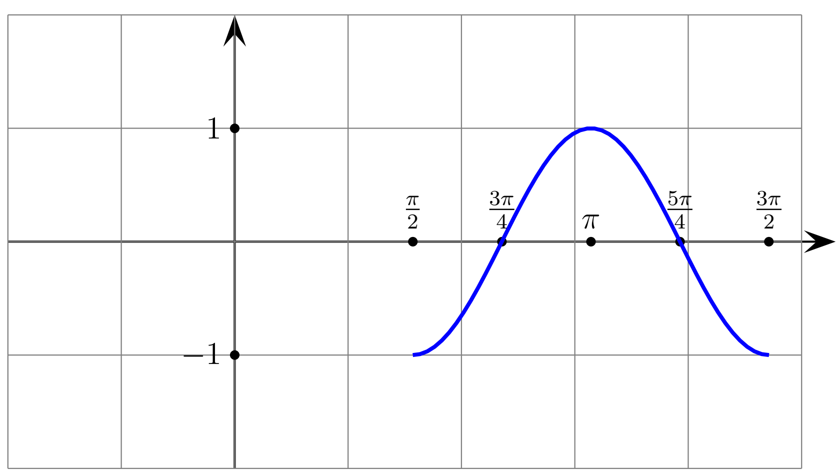


□

7. Sketch a complete cycle of $y = -\cos(2x - \pi)$

Solution. The frequency of the function is 2 and therefore its period is $\frac{2\pi}{2} = \pi$. The altitude is 1 and the starting point (phase shift) is given by $2x - \pi = 0$, that is, $x = \frac{\pi}{2}$. Also due to the negative sign the function will start at $y = -1$. A quarter of the period is $\frac{\pi}{4}$, so we construct a table of five values starting at the starting point $\frac{\pi}{2}$ and incrementing the x value by $\frac{\pi}{4}$, four times:

x	y
$\frac{\pi}{2}$	-1
$\frac{3\pi}{4}$	0
π	1
$\frac{5\pi}{4}$	0
$\frac{3\pi}{2}$	-1



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