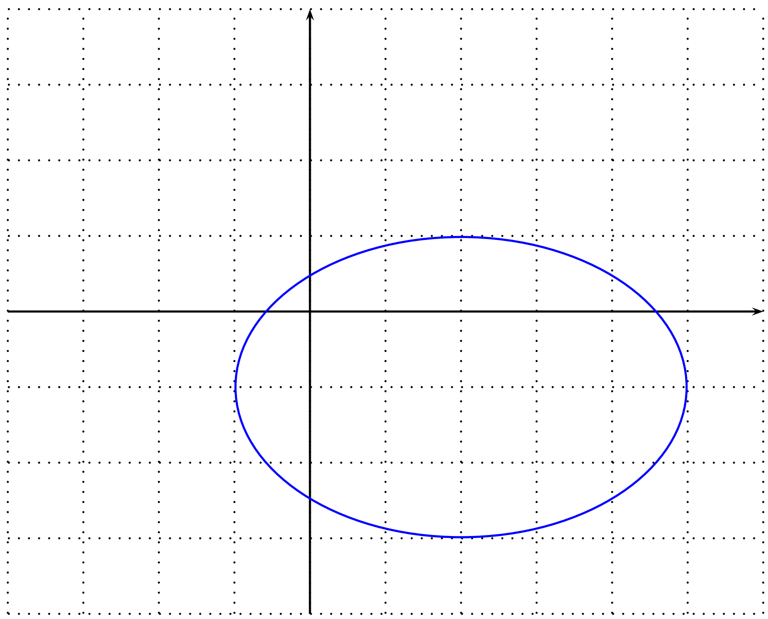
Second Exam

Take home

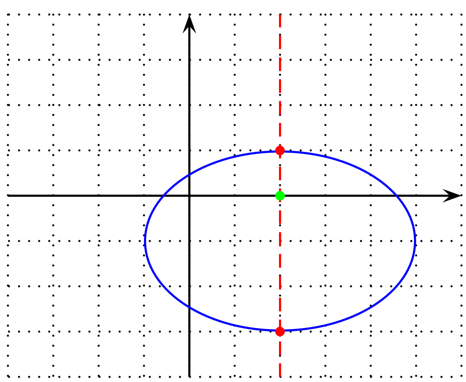
Due: April 28

1. The graph of the ellipse
$$\frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} = 1$$
is shown bellow:



(a) Explain why this is not the graph of a function.

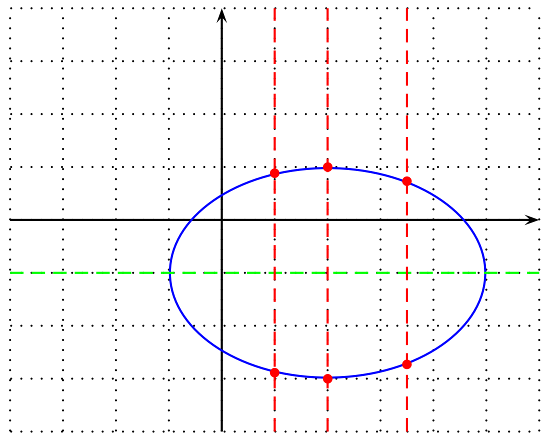
Solution. This is not the graph of a function because there are values of x that are related to two different values of y – actually any value of x in the interval $(-1, 5)$ is related to two different values of y. We illustrate this for $x = 2$: the vertical line $x = 2$ intersects the graph in two different points.



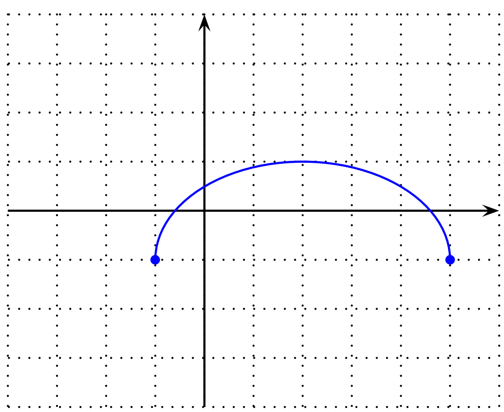
□

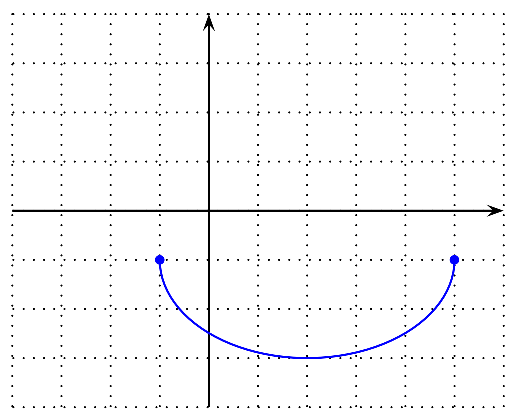
(b) How can we restrict the range so that we obtain a function?

Solution. As described above the reason that this relation is not a function is that there are two different values of y that correspond to a given value of x in the interval $(-1, 5)$. Actually from the graph we see that these two values of y are symmetric with respect to the horizontal line $y = -1$. We illustrate this bellow for three values of x ($x = 1, 2, 3.5$).



From the above it follows that if we take only the part of the graph that is above (or only the part that is bellow) the line $y = -1$ we will get the graph of a function. This means restricting the range to be either $[-1, 2]$ or $[-3, 1]$. The graphs of the functions obtained by these choices are shown bellow:

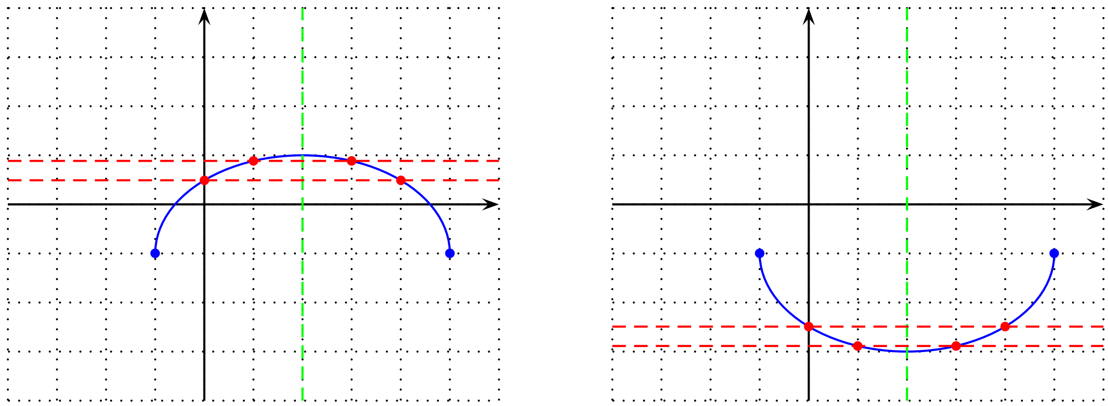




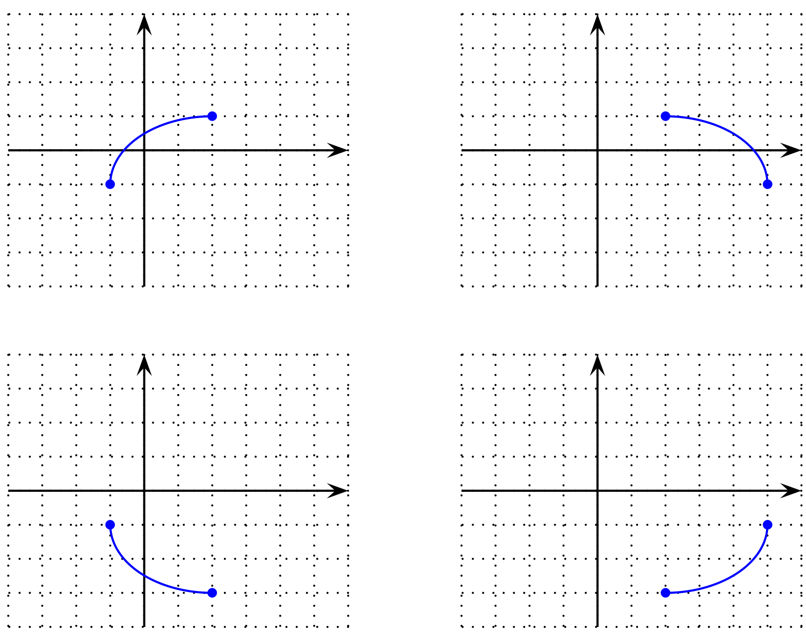
□

(c) Is the function you obtained in the previous step one-to-one? If not how can you restrict the domain so that it becomes one-to-one?

Solution. None of the two functions we obtain in the previous question is one-to-one. For, there are values of y that they are related to different values of x. Actually, from the picture we can see that if two x are symmetric with respect to the vertical line $x = 2$ their corresponding values of y are equal, and this is the only way that two values of x are related to the same value of y. We illustrate this in the following picture, $x = 1$ and $x = 3$ have the same y-value, and so do $x = 0$ and $x = 4$:



From the above discussion it follows that if we restrict the domain of either function either to the left or to the right of $x = 2$, we will get an one-to-one function. So we should restrict the domain to either $[-1, 2]$ or to $[2, 5]$. These two choices are shown bellow for each of the functions obtained in the previous question;



□

2. Let $f(x) = \dfrac{2}{x-1}$ and $g(x) = \dfrac{3}{x}$. Find $f \circ g$. Your answer should include the domain as well as the formula.

Solution. We first find the formula for $f \circ g$:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= \frac{2}{\frac{3}{x} - 1} \\ &= \frac{2x}{3 - x}\end{aligned}$$

In order for x to be in the domain of $f \circ g$ the following two conditions must hold:

1. x is in the domain of g, and
2. $g(x)$ is in the domain of f.

The first condition gives that $x \neq 0$. The second condition means that the formula of $f \circ g$ is defined for x, which means that $x \neq 3$. Putting these two conditions together we have that the domain of $f \circ g$ is $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$. □

3. Prove that $f(x) = \frac{2x - 5}{3x + 2}$ and $g(x) = -\frac{2x + 5}{3x - 2}$ are a pair of inverse functions.

Solution. We need to show that the following two conditions hold:

1. for x in the domain of g, $f(g(x)) = x$, and
2. for x in the domain of f, $g(f(x)) = x$.

For the first condition we have:

$$\begin{aligned}f(g(x)) &= \frac{2 \cdot \left(-\frac{2x + 5}{3x - 2}\right) - 5}{3 \cdot \left(-\frac{2x + 5}{3x - 2}\right) + 2} \\ &= \frac{-2(2x + 5) - 5(3x - 2)}{-3(2x + 5) + 2(3x - 2)} \\ &= \frac{-4x - 10 - 15x + 10}{-6x - 15 + 6x - 4} \\ &= \frac{-19x}{-19} \\ &= x\end{aligned}$$

And for the second condition we have:

$$
\begin{aligned}
g\left(f(x)\right) &= -\frac{2\cdot\dfrac{2x-5}{3x+2}+5}{3\cdot\dfrac{2x-5}{3x+2}-2} \\
&= -\frac{2(2x-5)+5(3x+2)}{3(2x-5)-2(3x+2)} \\
&= -\frac{4x-10+15x+10}{6x-15-6x-4} \\
&= -\frac{19x}{-19} \\
&= x
\end{aligned}
$$

□

4. Let $f(x)=\sqrt{x+1}$ and $g(x)=x^2-1$. Are f and g a pair of inverse functions? Justify your answer.

   Solution. They are not a pair of inverse functions. For, it is easy to see that $g(x)$ is not one-to-one so it doesn't have an inverse function. Indeed, $g(-x)=(-x)^2-1=x^2-1=g(x)$ so that g is takes the same value at opposite values of x (i.e. g is an even function). □

5. Sketch a graph of each of the following functions. The graph should correctly reflect end behavior, x and y intercepts, and possible asymptotes:

   (a) $f(x)=-x^3+4x^2+11x-30$

   Solution. The end behavior of a polynomial function is determined by its leading term. In our case the leading term is $-x^3$ so we have that $f(x)\to\infty$ as $x\to-\infty$, and $f(x)\to-\infty$ as $x\to\infty$.

   The y-intercept is obtained by substituting $x=0$ in the formula defining f. So the y–intercept is at $(0,-30)$.

   To find the x-intercepts of the graph of $y=f(x)$ we have to solve the equation

   $$-x^3+4x^2+11x-30=0$$

   We'll try to find rational solutions. According to the Rational Zero Theorem the only possible rational roots of this equation are the divisors of its constant term -30. So the only possible rational roots are $\{\pm1,\pm2,\pm3,\pm5,\pm6,\pm10,\pm15,\pm30\}$.

   We first try $x=1$:

   $$
   \begin{array}{r|rrrr}
    & -1 & 4 & 11 & -30 \\
   1 & & -1 & 3 & 14 \\
   \hline
    & -1 & 3 & 14 & -16
   \end{array}
   $$

So $x = 1$ is not a root. we then try $x = -1$:

$$\begin{array}{r|rrrr} & -1 & 4 & 11 & -30 \\ -1 & & 1 & -5 & -6 \\ \hline & -1 & 5 & 6 & -36 \end{array}$$

Also not a root. Next we try $x = 2$:

$$\begin{array}{r|rrrr} & -1 & 4 & 11 & -30 \\ 2 & & -2 & 4 & 30 \\ \hline & -1 & 2 & 15 & 0 \end{array}$$

So $x = 2$ is a root and the quotient by $x - 2$ is $-x^2 + 2x + 15$. The quotient is a quadratic polynomial so we can use the quadratic formula to find the roots. The discriminant is $D = 2^2 - 4(-1)15 = 64$ so the roots are

$$x = \frac{-2 \pm \sqrt{64}}{-2} = -\frac{-2 \pm 8}{2} = \begin{cases} -3 \\ 5 \end{cases}$$

So the x-intercepts are $(-3, 0), (2, 0), (5, 0)$. And since all the roots are single the graph of the function will cut through the x-axis at all the intersection points.

This information is enough to allow us to draw a sketch of the graph of $y = f(x)$: see Figure 1 □

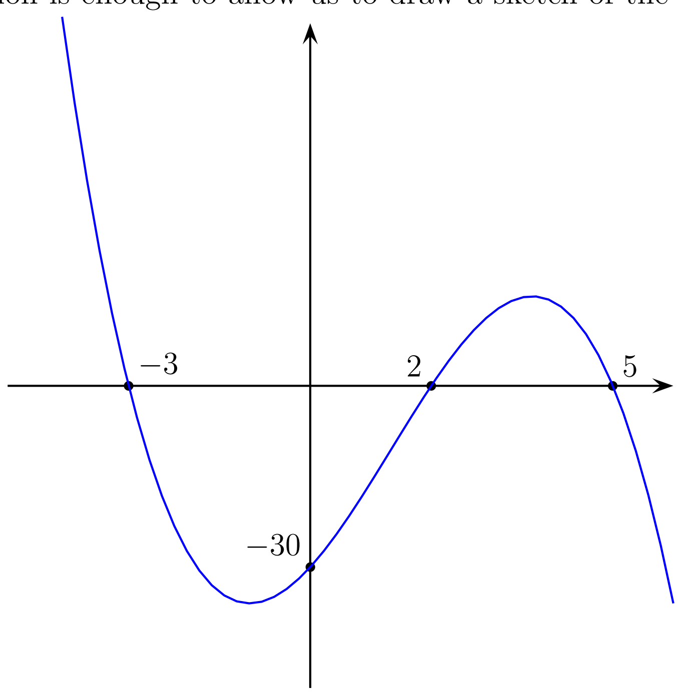


Figure 1: The graph of $y = -x^3 + 4x^2 + 11x - 30$

(b) $g(x) = \dfrac{2x+4}{x^2-3x-18}$

Solution. This is a rational function. We first find any vertical and horizontal asymptotes:

- Since the degree of the denominator is larger than the degree of the numerator the graph will have the x–axis as a horizontal asymptote.
- Vertical asymptotes are located at the roots of the denominator:

$$x^2 - 3x - 18 = 0 \Longleftrightarrow (x-6)(x+3) = 0 \Longleftrightarrow x = 6 \text{ or } x = -3$$

  None of these roots are roots of the numerator, therefore the graph will have two vertical asymptotes at $x = -3$ and $x = 6$.

Next we determine x and y intercepts. The y-intercept is at $g(0) = -\dfrac{4}{-18} = \dfrac{2}{9}$. The x intercepts are located at the roots of the numerator, so we have only one x intercept at $x = -2$.

Next we determine how the sign of $g(x)$ changes. The x intercept and the points where the function is undefined partition the x axis into four intervals $(-\infty, -3)$, $(-3, -2)$, $(-2, 6)$, $(6, \infty)$; and the sign of $g(x)$ will be constant in these intervals. So we pick one point in each of these intervals to determine the sign of $g(x)$ in that interval. We summarize the results in the next table:

	$-\infty$	-3	-2	6	∞
$g(x)$	$-$	$+$	$-$	$+$	

With this information we can sketch the graph of $y = g(x)$: see Figure 2 □

6. Find the domain of each of the following functions:

(a) $f(x) = \sqrt{\dfrac{x+3}{x-4}}$

Solution. In order for $f(x)$ to be defined we need:

1. the expression under the radical must be non-negative and,
2. the denominator has to be non-zero.
   We deal with the second condition first. We need $x - 4 \neq 0$, in other words,

$$x \neq 4 \tag{1}$$

We now deal with the first condition: we need to solve the inequality:

$$\frac{x+3}{x-4} \geq 0$$

This is a rational inequality. To solve it, we find the zeros and the point of vertical asymptotes. There is one zero at $x = -3$ and a vertical asymptote at

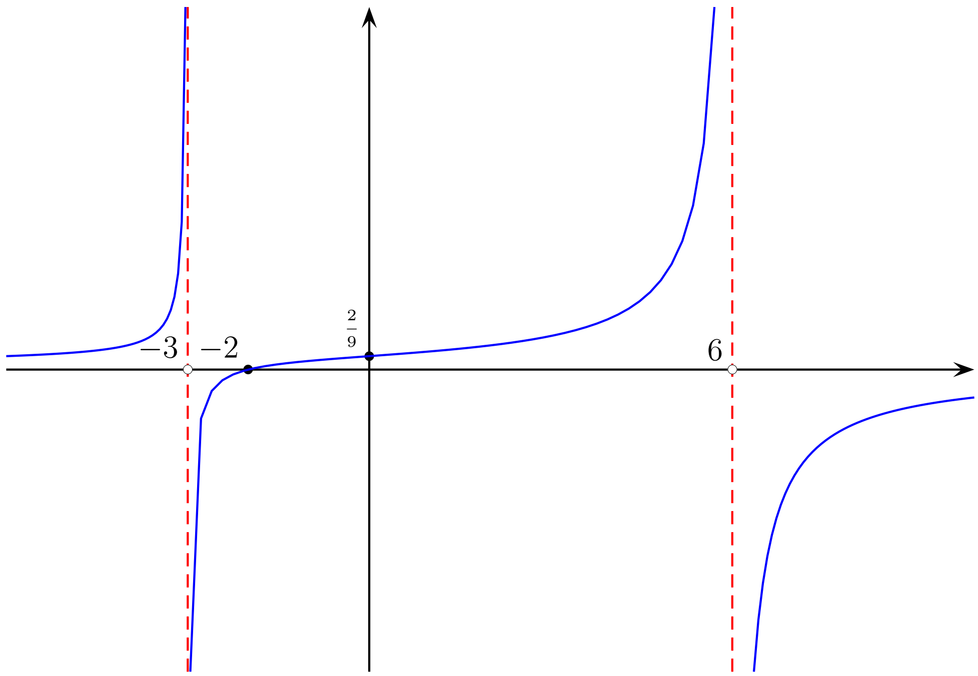
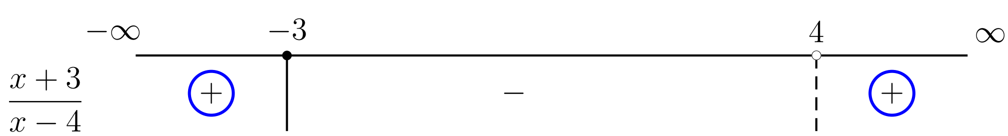


Figure 2: The graph of $y = -\dfrac{2x+4}{x^2-3x-18}$

$x = 4$. These break the real line into three intervals and the sign of the expression will be constant in each of these intervals. To determine the sign of the expression then we need to choose one point in each of the intervals $(-\infty, -3)$, $(-3, 4)$, $(4, \infty)$ and find the sign of the value of g at that point. For the first interval we can choose a "very negative number", e.g. $x = -1000000000$, and then both numerator and denominator will be negative so the expression is positive; for the second interval we choose $x = 0$ at which point the expression is $-\dfrac{3}{4}$; and for the third interval we can choose a very large positive number, e.g. $x = 1000000000000$ and then both numerator and denominator are positive so the expression is positive. We summarize all this in the following table:



From the table we see that the solution to our inequality is $(-\infty, -3] \cup (4, \infty)$. Since the condition (1) is also satisfied in this set we conclude that the domain of f is

$$(-\infty, -3] \cup (4, \infty)$$

□

(b) $g(x) = \ln(x^4 + 2x^3 - 16x^2 - 2x + 15)$

Solution. In order for $g(x)$ to be defined we need the argument of the logarithmic function to be positive, in other words the domain of $g(x)$ is the solution of the inequality:

$$x^4 + 2x^3 - 16x^2 - 2x + 15 > 0$$

To solve this inequality we first locate the zeros of the LHS: according to the rational zero theorem, the only possible rational zeros of the LHS are the divisors of 15, i.e, $\{\pm 1, \pm 3, \pm 5, \pm 15\}$. We try $x = 1$ first:

$$\begin{array}{r|rrrrr} & 1 & 2 & -16 & -2 & 15 \\ 1 & & 1 & 3 & -13 & -15 \\ \hline & 1 & 3 & -13 & -15 & 0 \end{array}$$

So $x = 1$ is a zero. We try $x = 1$ again to see whether it is a multiple zero:

$$\begin{array}{r|rrrr} & 1 & 3 & -13 & -15 \\ 1 & & 1 & 4 & -9 \\ \hline & 1 & 4 & -9 & -24 \end{array}$$

So $x = 1$ is a single zero. We try $x = -1$ next:

$$\begin{array}{r|rrrr} & 1 & 3 & -13 & -15 \\ -1 & & -1 & -2 & 15 \\ \hline & 1 & 2 & -15 & 0 \end{array}$$

So $x = -1$ is also a solution. The quotient from the last division is the quadratic polynomial $x^2 + 2x - 15$, which factors as $(x + 5)(x - 3)$. In sum we have that the LHS of the inequality factors as

$$x^4 + 2x^3 - 16x^2 - 2x + 15 = (x - 3)(x - 1)(x + 1)(x + 5)$$

Now we can construct the following table of signs:

	$-\infty$ to -5	-5 to -1	-1 to 1	1 to 3	3 to ∞
$p(x)$	⊕	$-$	⊕	$-$	⊕

Thus the domain of $g(x)$ is

$$(-\infty, -5) \cup (-1, 1) \cup (3, \infty)$$

□

7. Let $f(x) = e^{3x-5}$.

   (a) Find the inverse function f^{-1}.

   Solution. As a relation f is given by $y = e^{3x-5}$. To find the inverse relation we interchange x and y, so f^{-1} as a relation is given by the formula:

$$x = e^{3y-5}$$

   Now we solve for y to find a formula for $f^{-1}(x)$:

$$\begin{aligned} x = e^{3y-5} &\iff \ln x = \ln e^{3y-5} \\ &\iff \ln x = 3y - 5 \\ &\iff \ln x + 5 = 3y \\ &\iff \frac{\ln x + 5}{3} = y \end{aligned}$$

   Thus,

$$f^{-1}(x) = \frac{\ln x + 5}{3}$$

   □

   (b) Sketch both functions on the same coordinate system.

   Solution. The graph of $y = f(x)$ is obtained by shifting the graph of $y = e^{3x}$ to the right by 5 units. The graph of $y = f^{-1}(x)$ is obtained by first sifting the graph of $y = \ln x$ up by 5 units and then scaling by $\frac{1}{3}$. Of course, the graphs are symmetric with respect to the line $y = x$. See Figure 3 □

8. Suppose $\log_5 a = 4$, $\log_5 b = 3$ and $\log_5 c = -2$. Evaluate the following expression:

$$\log_5\left(\frac{25b^3\sqrt{a}}{c^5}\right)$$

Solution. We first expand the given expression, and then substitute the given values:

$$\begin{aligned} \log_5\left(\frac{25b^3\sqrt{a}}{c^5}\right) &= \log_5 25b^3\sqrt{a} - \log_5 c^5 \\ &= \log_5 25 + \log_5 b^3 + \log_5 \sqrt{a} - 5\log_5 c \\ &= 2 + 3\log_5 b + \frac{1}{2}\log_5 a - 5\log_5 c \\ &= 2 + 3\cdot 3 + \frac{1}{2}\cdot 4 - 5\cdot(-2) \\ &= 2 + 9 + 2 + 10 \\ &= 23 \end{aligned}$$

□

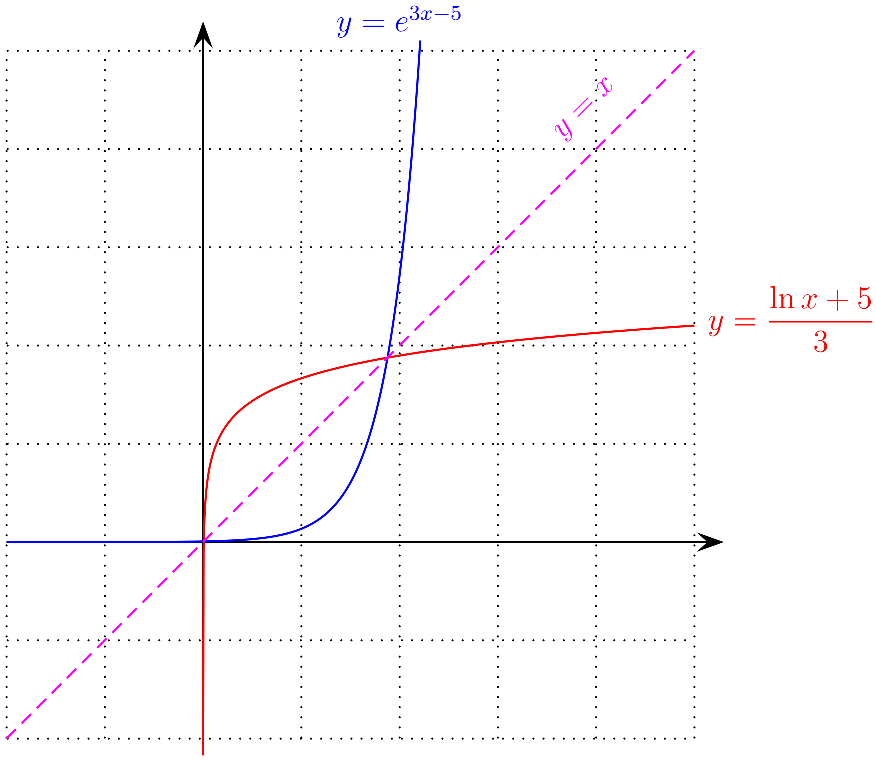


Figure 3: The graphs of the two inverse functions in Question 7b

9. Solve $\log_2(2x+8) - \log_2(x-3) = 4$

Solution. We first find the domain of the functions involved to see when the equation is defined: all expressions that appear as arguments to logarithms have to be positive, so we need $2x+8 > 0$ and $x-3 > 0$. The first condition is equivalent to $x > -4$ and the second to $x > 3$. The equation is therefore defined when $x > 3$, that is in the interval $(3, \infty)$.

Now we solve by first contracting and then exponentiating both sides:

$$\begin{aligned}
\log_2(2x+8) - \log_2(x-3) = 4 &\iff \log_2 \frac{2x+8}{x-3} = 4 \\
&\iff 2^{\log_2 \frac{2x+8}{x-3}} = 2^4 \\
&\iff \frac{2x+8}{x-3} = 16 \\
&\iff 2x+8 = 16(x-3) \\
&\iff 2x+8 = 16x - 48 \\
&\iff 56 = 14x \\
&\iff \frac{56}{14} = x \\
&\iff 4 = x
\end{aligned}$$

The solution lies in the interval that the equation is defined so it is accepted. □

10. Sketch two full cycles of the graph of $y = 3\sin 2x$.

Solution. We have the following table of values:

x	$2x$	$3\sin 2x$
0	0	0
$\frac{\pi}{4}$	$\frac{\pi}{2}$	3
$\frac{\pi}{2}$	π	0
$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	-3
π	2π	0
$\frac{5\pi}{4}$	$\frac{5\pi}{2}$	3
$\frac{3\pi}{2}$	3π	0
$\frac{7\pi}{4}$	$\frac{7\pi}{2}$	-3
2π	4π	0

So we get the graph of Figure 4

□

11. **Extra Credit:** Given that the remainder of the division

$$\frac{x^4 - 2x^3 + 5x^2 + 10x - 20}{x - \sqrt{5}}$$

is 30, solve the following equation:

$$x^4 - 2x^3 + 5x^2 + 10x - 50 = 0$$

Solution. We setup notation first: let $p(x) = x^4 - 2x^3 + 5x^2 + 10x - 20$ and $q(x) = x^4 - 2x^3 + 5x^2 + 10x - 50$. Then we have to solve the equation

$$q(x) = 0$$

Notice that $q(x) = p(x) - 30$, so since the remainder of the division of $p(x)$ by $x - \sqrt{5}$ is 30, $x - \sqrt{5}$ divides $q(x)$. Therefore $q(\sqrt{5}) = 0$, in other words $\sqrt{5}$ is root of $q(x)$. Since

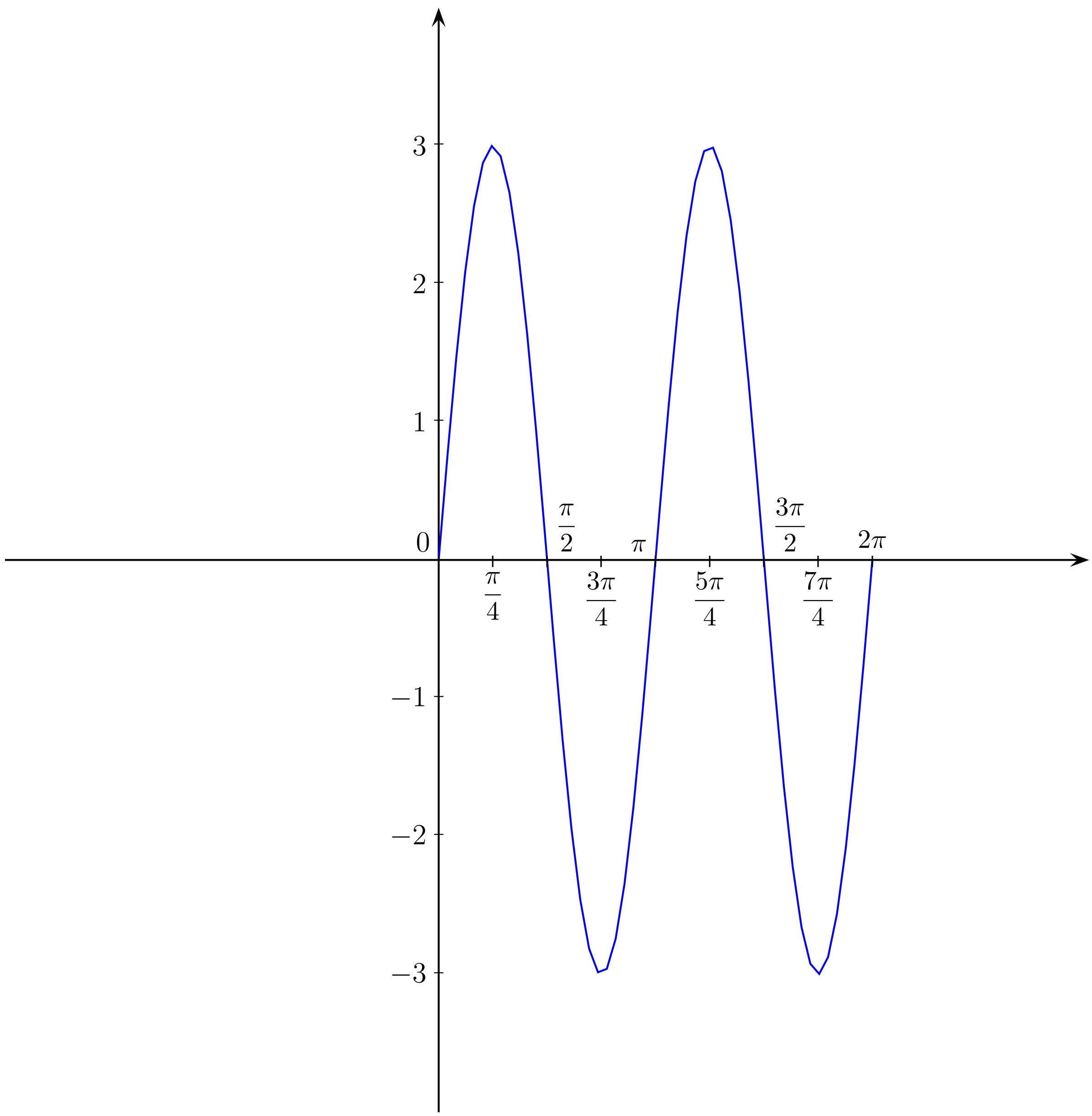


Figure 4: Two cycles of $y = 3\sin 2x$

$q(x)$ has integer coefficients $-\sqrt{5}$ is a root of $q(x)$ as well. It follows that $(x-\sqrt{5})(x+\sqrt{5})$ is a factor of $q(x)$. Now $(x-\sqrt{5})(x+\sqrt{5}) = x^2 - 5$, so we perform the division:

$$\begin{array}{rrrrrl} x^4 & -2x^3 & +5x^2 & +10x & -50 & = \left(x^2-5\right)\left(x^2-2x+10\right) \\ -x^4 & & +5x^2 & & & \\ \hline & -2x^3 & +10x^2 & +10x & & \\ & 2x^3 & & -10x & & \\ \hline & & 10x^2 & & -50 & \\ & & -10x^2 & & +50 & \\ \hline & & & & 0 & \end{array}$$

So

$$q(x) = 0 \Longleftrightarrow x^2 - 5 = 0 \text{ or } x^2 - 2x + 10 = 0$$

Thus we need to solve $x^2 - 2x + 10 = 0$. This is a quadratic equation with discriminant $D = (-2)^2 - 4 \cdot 1 \cdot 10 = -36$ so its solutions are

$$x = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

In sum, we have the following four solutions:

$$x = \sqrt{5}, \quad x = -\sqrt{5}, \quad x = 1 + 3i, \quad x = 1 - 3i.$$

□