Solutions to Practice Exam

1. Subtract. Simplify your answer as much as possible: $\dfrac{3}{x-5} - \dfrac{18}{x^2-2x-15}$

   Solution. The LCD is $x^2 - 2x - 15 = (x+3)(x-5)$. So we have:

$$\begin{aligned}
\frac{3}{x-5} - \frac{18}{x^2-2x-15} &= \frac{3(x+3)}{(x+3)(x-5)} - \frac{18}{(x+3)(x-5)} \\
&= \frac{3x+9}{(x+3)(x-5)} + \frac{18}{(x+3)(x-5)} \\
&= \frac{3x+9-18}{(x+3)(x-5)} \\
&= \frac{3x-9}{(x+3)(x-5)}
\end{aligned}$$

□

2. Divide. Simplify your answer as much as possible: $\dfrac{2b}{b-3} \div \dfrac{b^2+3b}{b^2-3b-18}$

   Solution. We have:

$$\begin{aligned}
\frac{2b}{b-3} \div \frac{b^2+3b}{b^2-3b-18} &= \frac{2b}{b-3} \cdot \frac{b^2-3b-18}{b^2+3b} \\
&= \frac{2b}{b-3} \cdot \frac{(b+3)(b-6)}{b(b+3)} \\
&= \frac{2b}{b-3} \cdot \frac{b-6}{b} \\
&= \frac{2}{b-3} \cdot (b-6) \\
&= \frac{2b-12}{b-3}
\end{aligned}$$

□

3. Solve: $\dfrac{3}{x+7} - \dfrac{5}{x^2+2\,x-35} = \dfrac{44}{x-5}$

   Solution. We first determine the values of x for which the equation is defined: we need $x+7 \neq 0$, $x^2+2x-35 \neq 0$, and $x-5 \neq 0$. Since $x^2+2x-35 = (x+7)(x-5)$, the equation is defined when $x \neq -7$ and $x \neq 5$.

   To solve the equation we multiply both sides with the LCD $(x+7)(x-5)$. We get:

$$\begin{aligned}
\frac{3}{x+7} - \frac{5}{x^2+2\,x-35} = \frac{44}{x-5} &\iff 3(x-5) - 5 = 44(x+7) \\
&\iff 3x - 15 - 5 = 44x + 308 \\
&\iff 3x - 20 = 44x + 308 \\
&\iff -328 = 41x \\
&\iff x = -8
\end{aligned}$$

   The equation is defined for $x = -8$ so the solution is accepted. □

4. Solve: $\sqrt{x-2}+8=x$

Solution. We first isolate the radical expression and then square both sides:

$$\begin{aligned}\sqrt{x-2}+8=x &\iff \sqrt{x-2}=x-8\\ &\implies \left(\sqrt{x-2}\right)^2=(x-8)^2\\ &\iff x-2=x^2-16x+64\\ &\iff 0=x^2-17x+66\end{aligned}$$

This is a quadratic equation with discriminant $D=(-17)^2-4\cdot 66=25=5^2$. So it has two solutions:

$$x=\frac{17\pm 5}{2}=\begin{cases}11\\ \\ 6\end{cases}$$

We now check whether these are solutions of the original equation. We first check $x=11$:

$$\sqrt{11-2}+8=11 \iff \sqrt{9}+8=11 \iff 3+8=11$$

which is true. So $x=11$ is a solution.

We now check $x=6$:

$$\sqrt{6-2}+8=6 \iff \sqrt{4}+8=6 \iff 2+8=6$$

which is false. So $x=6$ is not a solution. □

5. Simplify: $\dfrac{\dfrac{5}{x-3}-\dfrac{2}{x+3}}{\dfrac{3}{x^2-9}}$

Solution. We multiply both numerator and denominator of the fraction with the LCD of all denominators $(x-3)(x+3)$ and then simplify as much as possible:

$$\begin{aligned}\frac{\dfrac{5}{x-3}-\dfrac{2}{x+3}}{\dfrac{3}{x^2-9}} &= \frac{5(x+3)-2(x-3)}{3}\\ &=\frac{5x+15-2x+6}{3}\\ &=\frac{3x+21}{3}\\ &=\frac{3(x+7)}{3}\\ &=x+7\end{aligned}$$

□

6. Graph the parabola $x=y^2-4y+3$. Your graph should correctly indicate the vertex, the axis of symmetry, the y-intercepts, the x-intercept and the point symmetric to the x-intercept.

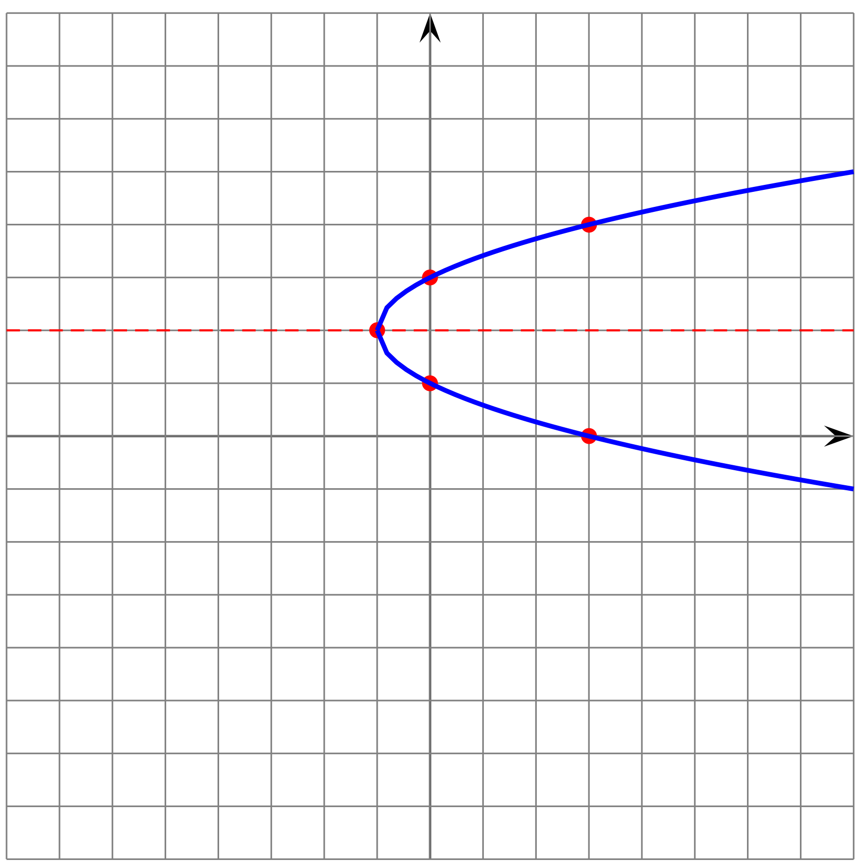
Solution. The parabola will "open along the x-axis". We first find its vertex by completing the square:

$$\begin{aligned} x = y^2 - 4y + 3 &\iff x + 4 = y^2 - 4y + 4 + 3 \\ &\iff x + 4 = (y-2)^2 + 3 \\ &\iff x = (y-2)^2 + 3 - 4 \\ &\iff x = (y-2)^2 - 1 \end{aligned}$$

So the vertex is at $(1, 2)$ and the axis of symmetry is $y = 2$. We then find four additional points in the parabola by letting y be one unit more, one unit less, two units more, and two unit less than the y of the vertex and computing the corresponding values of x.

x	y
-1	2
0	3
0	1
3	4
3	0

The x-intercept is the point in the parabola that has $y = 0$. This has already been computed in the table above. So the x-intercept is $(3, 0)$ and the point symmetric to it is $(3, 4)$.



□

7. Solve: $2x^2 - 3 = 6x$

Solution. This is a quadratic equation. We first transfer all the terms to the LHS and then solve using the quadratic formula:

$$2x^2 - 3 = 6x \Longleftrightarrow 2x^2 - 6x - 3 = 0$$

The discriminant is $D = (-6^2 - 4 \cdot 2 \cdot (-3)) = 36 + 24 = 60$ So we have two solutions:

$$x = \frac{6 \pm \sqrt{60}}{2 \cdot 2} = \frac{6 \pm 2\sqrt{15}}{4} = \frac{3 \pm \sqrt{15}}{2}$$

□

8. Find the center and radius of the circle with equation $y^2 - 4\,y + x^2 + 2\,x = -2$

Solution. We put the equation in standard form by completing the terms involving x and the terms involving y into squares:

$$y^2 - 4\,y + x^2 + 2\,x = -2 \Longleftrightarrow y^2 - 4\,y + +4 + x^2 + 2\,x + 1 = -2 + 4 + 1$$
$$(x+1)^2 + (y-2)^2 = 3$$

So the center of the circle is at $(-1, 2)$ and its radius is $\sqrt{3}$. □

9. Simplify: $\left(\frac{-8x^{-20}y^{13}}{xy^4}\right)^{-\frac{1}{3}}$. Assume all variables represent positive numbers. The answer should contain only positive integers as exponents.

Solution. We first simplify the expression inside the parenthesis and then we use the properties of exponents to simplify:

$$\begin{aligned}
\left(\frac{-8x^{-20}y^{13}}{xy^4}\right)^{-\frac{1}{3}} &= \left(-2^3x^{-20}y^{13}x^{-1}y^{-4}\right)^{-\frac{1}{3}} \\
&= \left(-2^3x^{-21}y^9\right)^{-\frac{1}{3}} \\
&= -2^{-1}x^7y^{-3} \\
&= \frac{x^7}{2y^3}
\end{aligned}$$

□

10. Divide: $\frac{15i - 5}{3 + i}$. Express your answer in the form $a + bi$ where a and b are real numbers.

Solution.

$$\begin{aligned}\frac{15i-5}{3+i} &= \frac{15i-5}{3+i}\cdot\frac{3-i}{3-i}\\ &= \frac{45i-15i^2-15-5i}{9-i^2}\\ &= \frac{45i+15-15-5i}{9+1}\\ &= \frac{40i}{10}\\ &= 4i\end{aligned}$$

□

11. Find an equation of the line tangent of the circle $(x-1)^2+(y+2)^2=4$ at the point $(3,-2)$.

    Solution. The tangent to a circle at a point is perpendicular to the radius at that point. The given circle has center $(1,-2)$, so the radius to the point $(3,-2)$ has slope $\frac{-2-(-2)}{3-1}=0$. So the radius is horizontal therefore the tangent will be vertical. So the tangent is a vertical line passing through the point $(3,-2)$, and thus its equation is

    $$x=3$$

    □

12. Determine the equation of the parabola with focus $(0,2)$ and directrix $y=-4$.

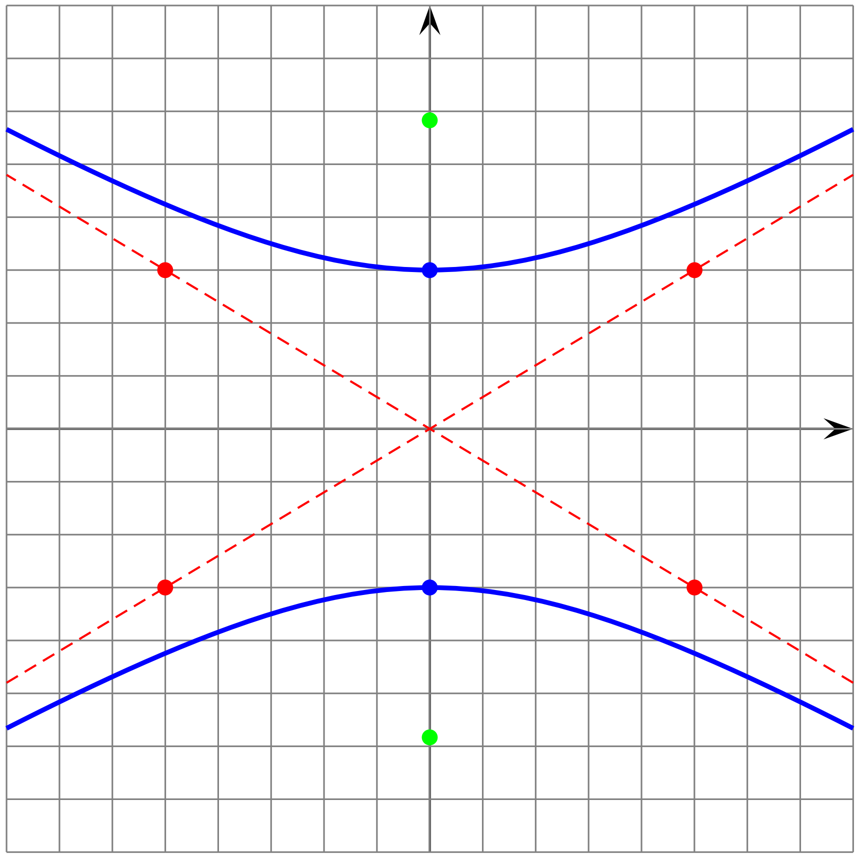
    Solution. The parabola is the set of points equidistant from the focus and the directrix. The square of the distance of a point with coordinates (x,y) from the point $(0,2)$ is $x^2+(y-2)^2$ while its distance from the line $y=-4$ is $(y+4)^2$. So the equation of the parabola with focus $(0,2)$ and directrix $y=-4$ is

    $$\begin{aligned}x^2+(y-2)^2=(y+4)^2 &\iff x^2+y^2-4y+4=y^2+8y+16\\ &\iff x^2-4y+4=8y+16\\ &\iff x^2-12=12y\\ &\iff \frac{x^2}{12}-1=y\end{aligned}$$

    □

13. Graph the hyperbola with equation $\frac{y^2}{9}-\frac{x^2}{25}=1$. The graph should correctly indicate the vertices, the foci and the asymptotes of the hyperbola.

    Solution. Since the positive term in the LHS is the one with y^2 the parabola "opens along the y-axis". Now, $a=5$, and $b=3$, so the vertices are at $(0,-3)$ and $(0,3)$. The asymptotes are $y=\pm\frac{3}{5}x$, and they are the diagonals of the rectangle with corners at $(3,5)$, $(-3,5)$, $(-3,-5)$ and $(3,-5)$. The foci are located at $(0,\pm c)$, where $c=\sqrt{a^2+b^2}$, so the foci are at $(0,\pm\sqrt{34})$. Using a calculator we see that $\sqrt{34}\approx 5.83095189485$. In sum we have the following graph:



□

14. Find the common points of the line $y = x + 4$ and the conic section $2\,y^2 - 12\,y + x^2 + 2\,x = -16$.

Solution. We need to solve the system:

$$\begin{cases} 2\,y^2 - 12\,y + x^2 + 2\,x &= -16 \\ y &= x + 4 \end{cases}$$

We substitute $x + 4$ for y in the first equation and solve:

$$\begin{aligned} 2\,(x+4)^2 - 12\,(x+4) + x^2 + 2\,x = -16 &\iff 2\,(x^2 + 8x + 16) - 12\,x + 48 + x^2 + 2\,x = -16 \\ &\iff 2x^2 + 16x + 32 - 12\,x - 48 + x^2 + 2\,x = -16 \\ &\iff 3x^2 + 6x - 16 = -16 \\ &\iff 3x^2 + 6x = 0 \\ &\iff x^2 + 3x = 0 \\ &\iff x(x+3) = 0 \\ &\iff x = 0 \text{ or } x = -3 \end{aligned}$$

Substituting these values of x in the second equation we find that the corresponding values of y are $y = 4$ and $y = 1$. Thus there are two common points: $(0, 4)$ and $(-3, 1)$. □

15. Simplify:

(a) $5\sqrt{12} - \sqrt{200} + 3\sqrt{18}$

Solution.

$$\begin{aligned} 5\sqrt{12} - \sqrt{200} + 3\sqrt{18} &= 5 \cdot 2\sqrt{3} - 10\sqrt{2} + 3 \cdot 3\sqrt{2} \\ &= 10\sqrt{3} - 10\sqrt{2} + 9\sqrt{2} \\ &= 10\sqrt{3} - \sqrt{2} \end{aligned}$$

□

(b) $\left(1 - \sqrt{3}\right)^2$

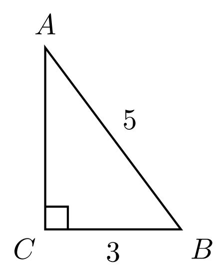
Solution.

$$\begin{aligned} \left(1 - \sqrt{3}\right)^2 &= 1 - 2\sqrt{3} + (\sqrt{3})^2 \\ &= 1 - 2\sqrt{3} + 3 \\ &= 4 - 2\sqrt{3} \end{aligned}$$

□

16. In a right triangle ABC we have $C = 90°$, $a = 3$, and $c = 5$ inches. Solve the triangle.

Solution. We have the following picture:



Using the Pythagorean theorem we get

$$3^2 + b^2 = 5^2 \iff b^2 = 25 - 9 \iff b^2 = 16 \iff b = 4$$

We now have:

$$\tan A = \frac{3}{4} = 0.75$$

Using a calculator we see that $\tan^{-1} 0.75 \approx 36.8698976458°$, so rounding to the nearest hundredth of a degree we have $A \approx 36.87°$.

We therefore have $B = 90° - A \approx 53.13°$. □

17. Find the exact value of each:

(a) $\tan 270° \cdot \sin 225°$

Solution. $\tan 270°$ is undefined because $270° - 180° = 90°$ and $\tan 90°$ is undefined. So the expression is undefined. □

(b) $\cos 1200°$

Solution. We first find an angle between $0°$ and $360°$ coterminal with $1200°$. Dividing $1200°$ by $360°$ we see that $1200° = 3 \cdot 360° + 120°$, so $1200°$ is coterminal with $120°$. So we need to find $\cos 120°$.

$120°$ is in the second quadrant and $180° - 120° = 60°$. So $\cos 120° = -\cos 60° = -\frac{1}{2}$. Thus:

$$\cos 1200° = -\frac{1}{2}$$

□

18. Find all angles θ, between $0°$ and $360°$ with $\sin\theta = -0.656$. Round your answers to the nearest tenth of a degree.

Solution. Using a calculator we see that $\sin^{-1}(-0.656) \approx -40.99551881° \approx -41°$. This solution is not between $0°$ and $360°$ but we can use it to get the two solutions between $0°$ and $360°$.

The first solution is coterminal to $-41°$: $\theta \approx 360° - 41° = 319°$.

The second solution is $\theta \approx 180° - (-41°) = 221°$. □

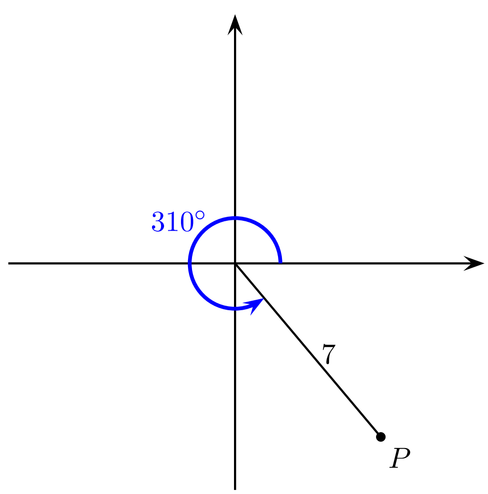
19. The angle of elevation from a sailboat in a lake to the top of a cliff is $75°$. The sailboat is 300 feet from the foot of the cliff. How high is the cliff? Round your answer to the nearest foot.

Solution. If h is the height of the cliff we have:

$$\tan 75° = \frac{h}{300} \Longrightarrow h = 300 \tan 75° \approx 1119.61524227$$

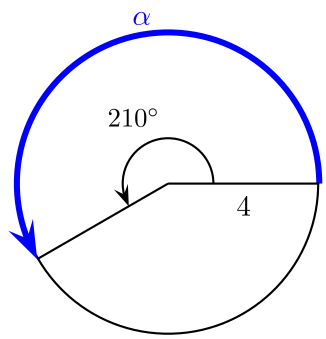
So, rounded to the nearest foot, the height of the cliff is 1120 ft. □

20. (a) Find the coordinates of the point P. Round your answer to the nearest hundredth.



Solution. We have $x = 7\cos 310° \approx 4.4995132678 \approx 4.5$ and $y = 7\sin 310° \approx -5.36231110183 \approx -5.36$. □

(b) Find the length of the arc α, where the corner of the angle is at the center of the circle. Give an exact answer.



Solution. We have that $\dfrac{210^\circ}{360^\circ} = \dfrac{7}{12}$. So α is $\dfrac{7}{12}$ of the whole circle. Now since the circle has radius 4 its length is $2 \cdot 4\pi = 8\pi$. So the length of α is

$$\frac{7}{12}(8\pi) = \frac{14\pi}{3}$$

□