Exam 5
The answers

1. Solve: $x^2 - 6x + 10 = 0$.

   Solution. This is a quadratic equation. Its discriminant is $D = (-6)^2 - 4 \cdot 1 \cdot 10 = -4$. So we get two (complex) solutions:

   $$x = \frac{6 \pm \sqrt{-4}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i$$

   □

2. Simplify: $\sqrt{500} - 3\sqrt{20} - 2\sqrt{45} + 2\sqrt{5}$.

   Solution.

   $$\begin{aligned}\sqrt{500} - 3\sqrt{20} - 2\sqrt{45} + 2\sqrt{5} &= 10\sqrt{5} - 3 \cdot 2\sqrt{5} - 2 \cdot 3\sqrt{5} + 2\sqrt{5} \\ &= 10\sqrt{5} - 6\sqrt{5} - 6\sqrt{5} + 2\sqrt{5} \\ &= 0\end{aligned}$$

   □

3. Simplify assuming all variables represent positive numbers. The answer should contain only positive integers as exponents.

   $$\left(\frac{4x^{-8}y^3}{z^{-6}}\right)^{-\frac{1}{2}}$$

   Solution.

   $$\begin{aligned}\left(\frac{4x^{-8}y^3}{z^{-6}}\right)^{-\frac{1}{2}} &= \frac{(4)^{-\frac{1}{2}}x^{-\frac{1}{2}\cdot(-8)}y^{-\frac{1}{2}\cdot 3}}{z^{-\frac{1}{2}(-6)}} \\ &= \frac{2^{-1}x^4y^{-\frac{3}{2}}}{z^3} \\ &= \frac{x^4}{2z^3y\sqrt{y}} \\ &= \frac{x^4}{2z^3y\sqrt{y}} \cdot \frac{\sqrt{y}}{\sqrt{y}} \\ &= \frac{x^4\sqrt{y}}{2z^3y^2}\end{aligned}$$

   □

4. Simplify. Express your answer in the form $a + bi$ where a and b are real numbers.

$$\frac{2i+4}{2i-1}$$

Solution.

$$\begin{aligned}\frac{2i+4}{2i-1} &= \frac{2i+4}{2i-1}\cdot\frac{2i+1}{2i+1}\\ &= \frac{-4+2i+8i+4}{-4-1}\\ &= \frac{10i}{-5}\\ \\ &= -2i\end{aligned}$$

□

5. Solve $x = \sqrt{x+7} - 5$.

Solution. We first isolate the radical expression and then square both sides:

$$\begin{aligned}x = \sqrt{x+7} - 5 &\Longleftrightarrow x + 5 = \sqrt{x+7}\\ &\Longrightarrow (x+5)^2 = \left(\sqrt{x+7}\right)^2\\ &\Longleftrightarrow x^2 + 10x + 25 = x + 7\\ &\Longleftrightarrow x^2 + 9x + 18 = 0\\ &\Longleftrightarrow (x+6)(x+3) = 0\\ &\Longleftrightarrow x = -6 \text{ or } x = -3\end{aligned}$$

We now check the two solutions: first $x = -6$:

$$-6 = \sqrt{-6+7} - 5 \Longleftrightarrow -6 = \sqrt{1} - 5 \Longleftrightarrow -6 = -4$$

So $x = -6$ is not a solution. We now check $x = -3$:

$$-3 = \sqrt{-3+7} - 5 \Longleftrightarrow -3 = \sqrt{4} - 5 \Longleftrightarrow -6 = 2 - 5$$

which is true. So the only solution is $x = -3$. □

6. Subtract: $\dfrac{x+4}{2x+10} - \dfrac{5}{x^2-25}$

Solution. The LCD of the two denominators is $2(x-5)(x+5)$. So we have:

$$\begin{aligned}\frac{x+4}{2x+10}-\frac{5}{x^2-25} &= \frac{(x+4)(x-5)}{2(x+5)(x-5)}-\frac{5\cdot 2}{2(x-5)(x+5)}\\ &= \frac{x^2-5x+4x-20-10}{2(x-5)(x+5)}\\ &= \frac{x^2-x-30}{2(x-5)(x+5)}\\ &= \frac{(x+5)(x-6)}{2(x-5)(x+5)}\\ &= \frac{x-6}{2(x-5)}\end{aligned}$$

□

7. Solve: $\dfrac{1}{x-1}-\dfrac{1}{x+1}=\dfrac{3x}{x^2-1}$.

   Solution. We first determine when the equation is defined. We need all denominators, $x-1$, $x+1$, and x^2-1 to be different than 0. So in order for the equation to be defined we need $x\neq 1$ and $x\neq -1$.

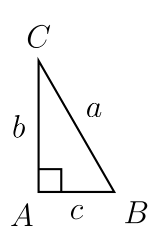
   Now we multiply both sides of the equation with the LCD $(x-1)(x+1)$, and solve the resulting equation:

$$\begin{aligned}\frac{1}{x-1}-\frac{1}{x+1}=\frac{3x}{x^2-1} &\iff (x+1)-(x-1)=3x\\ &\iff 2=3x\\ &\iff \frac{2}{3}=x\end{aligned}$$

   Since the equation is defined for $x=\dfrac{2}{3}$ we accept the solution. □

8. In a right triangle ABC we have $A=90^\circ$, $B=60^\circ$, and $a=4$ inches. Find c.

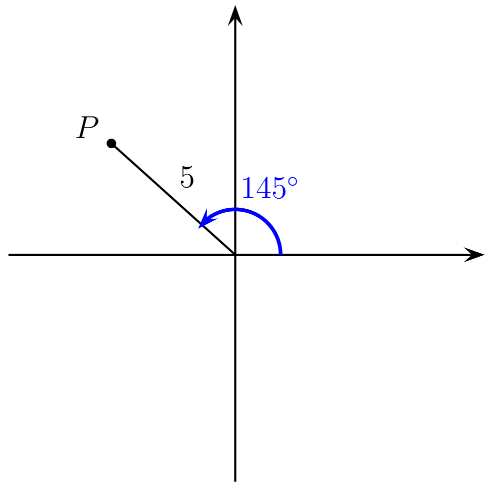
   Solution. The hypotenuse of the triangle is a and the side adjacent to B is c. So we have:



$$\begin{aligned}\cos B=\frac{c}{a} &\implies \cos 60^\circ=\frac{c}{4}\\ &\iff \frac{1}{2}=\frac{c}{4}\\ &\iff 2=c\end{aligned}$$

   Thus c is 2 inches. □

9. Find the coordinates of the point P. Round your answer to the nearest hundredth.



Solution. Let (x, y) be the coordinates of the point P. Then we have:

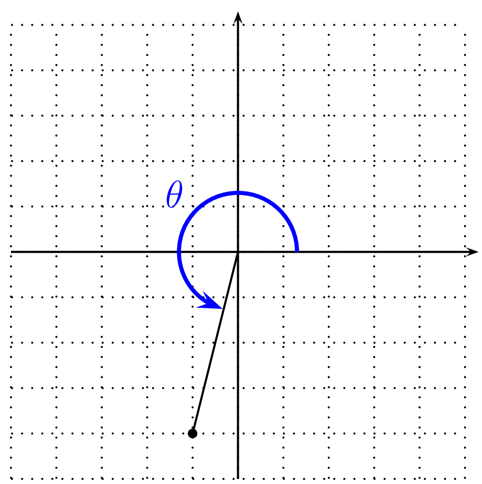
$$x = 5\cos 145° \approx 5(-0.819152044289) = -4.09576022145 \approx -4.1$$

and,

$$y = 5\sin 145° \approx 5 \cdot 0.573576436351 = 2.86788218176 \approx 2.87$$

□

10. Find the angle θ. Round your answer to the nearest tenth of a degree.



Solution. The endpoint of the segment shown in the figure has coordinates $(-1, -4)$. It follows that

$$\tan\theta = \frac{-4}{-1} = 4$$

Using a calculator we see that

$$\tan^{-1}(4) \approx 75.9637565321° \approx 76°$$

Since the terminal point of θ is in the third quadrant we have that $\theta \approx 180^\circ + 76^\circ = 256^\circ$ $\square$

11. Find all angles θ with $0^\circ \le \theta < 360^\circ$ so that $\cos\theta = -\dfrac{\sqrt{3}}{2}$.

    Solution. From the table of trigonometric values for special angles we know that $\cos 30^\circ = \dfrac{\sqrt{3}}{2}$. It follows then that the cosine of $180^\circ - 30^\circ = 150^\circ$ is $-\dfrac{\sqrt{3}}{2}$. So one of the solutions is $\theta = 150^\circ$. An other solution is $\theta = -150^\circ$, but this is not between 0° and 360°, so we take the coterminal angle $\theta = 360^\circ - 150^\circ = 210^\circ$. So we have two solutions $\theta = 150^\circ$ and $\theta = 210^\circ$. $\square$

12. A point different than $(0, 0)$ lies on the line $y = -2x$. What are the possible angles of reference for P?

    Solution. The slope of a line that passes through the origin, is the tangent of the angle that the line forms with the x-axis. Now the slope of $y = -2x$ is -2, so it follows that if θ is the angle of reference for P we have

    $$\tan\theta = -2$$

    This equations has two solutions between 0° and 360°, one in the second and one in the fourth quadrant. Using a calculator we see that $\tan^{-1}(-2) \approx -63.4349488229^\circ \approx -63.43^\circ$. Reference angles should be between 0° and 360°, so instead of the negative angle we take its coterminal angle $\theta \approx 360^\circ - 63.43^\circ = 296.57^\circ$.

    The solution in the second quadrant is $\theta \approx 180^\circ - 63.43^\circ = 116.57^\circ$. $\square$

13. Find the common points of the circle with equation $x^2 - 2x + y^2 + 6y = 40$ and the line with equation $y = 7x - 10$.

    Solution. To find the common points we have to solve the system:

    $$\begin{cases} x^2 - 2x + y^2 + 6y &= 40 \\ y &= 7x - 10 \end{cases}$$

    We substitute the second equation into the first, expand, simplify and solve:

    $$\begin{aligned} x^2 - 2x + (7x-10)^2 + 6(7x-10) = 40 &\iff x^2 - 2x + 49x^2 - 140x + 100 + 42x - 60 = 40 \\ &\iff 50x^2 - 100x + 40 = 40 \\ &\iff 50x^2 - 100x = 0 \\ &\iff x^2 - 2x = 0 \\ &\iff x(x-2) = 0 \\ &\iff x = 0 \text{ or } x = 2 \end{aligned}$$

    Substituting $x = 0$ into the second equation of the system gives $y = -10$, while substituting $x = 2$ gives $y = 4$. Thus there are two common points: $(0, -10)$ and $(2, 4)$. $\square$

14. Find the line that is tangent to the circle with equation $x^2 - 2x + y^2 - 6y = -8$ at the point $(2, 2)$.

Solution. The tangent line to a circle at a point is perpendicular to the radius of the circle at that point. We first find the center of the circle by putting the equation of the circle in standard form (by completing the squares):

$$\begin{aligned} x^2 - 2x + y^2 - 6y = -8 &\iff x^2 - 2x + 1 + y^2 - 6y + 9 = -8 + 1 + 9 \\ &\iff (x-1)^2 + (y-3)^2 = 2 \end{aligned}$$

So the center of the circle is at $(1, 3)$. Thus the radius to the point $(2, 2)$ has slope $\dfrac{3-2}{1-2} = -1$, so the tangent has slope 1. So the tangent of the circle at $(2, 2)$ is the line with slope 1 and passing through the point $(2, 2)$. Therefore its equation is:

$$y - 2 = x - 2 \iff y = x$$

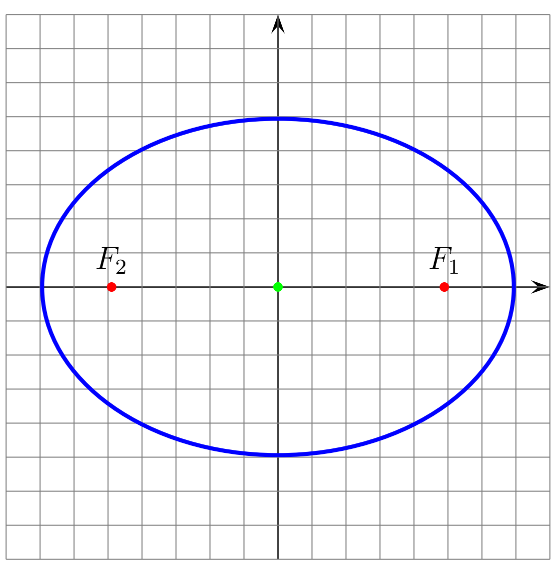
□

15. Sketch the graph of the ellipse

$$\frac{x^2}{49} + \frac{y^2}{25} = 1$$

The graph should correctly reflect the minor and major axis, the center and the foci.

Solution. The center is at $(0, 0)$. We have $a = \sqrt{49} = 7$ and $b = \sqrt{25} = 5$. Since a is larger the major axis is along the x-axis and the minor axis along the y-axis. The foci will be located in the x axis at the points $(\pm c, 0)$ where $c = \sqrt{a^2 - b^2}$. So $c = \sqrt{24} = 2\sqrt{6}$, and the foci are at $(\pm 2\sqrt{6}, 0)$. In order to plot this points we need to know an approximation of $2\sqrt{6}$. Using a calculator we find $2\sqrt{6} \approx 4.89897948556$.



□

16. Find the standard form of the equation of an ellipse with foci at $(-1, 0)$ and $(1, 0)$ given that the sum of the distances of a point of the ellipse from the tow foci is 4.

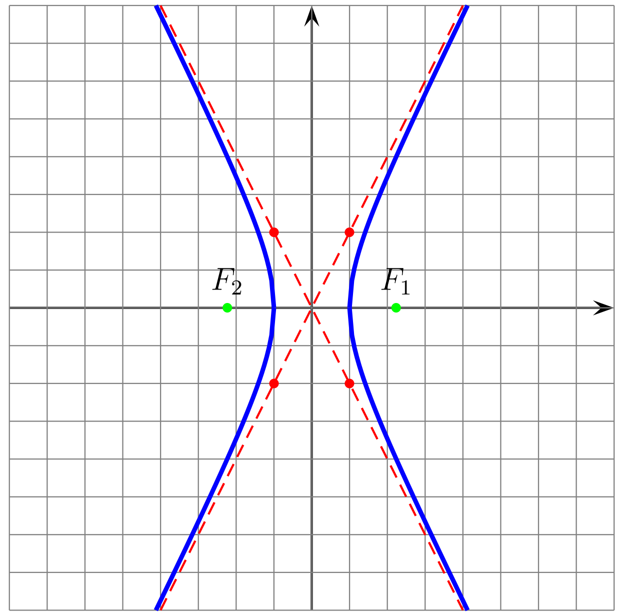
Solution. The ellipse is the set of points of the plane for which the sum of of the distances from the distance of the two foci is 4. The distance of a point with coordinates (x, y) from $(-1, 0)$ is given by $\sqrt{(x+1)^2 + y^2}$ and its distance from $(1, 0)$ is given by $\sqrt{(x-1)^2 + y^2}$. So the equation of the ellipse is:

$$\begin{aligned}
\sqrt{(x+1)^2+y^2}+\sqrt{(x-1)^2+y^2}=4 &\iff \sqrt{(x+1)^2+y^2}=4-\sqrt{(x-1)^2+y^2} \\
&\implies \left(\sqrt{(x+1)^2+y^2}\right)^2=\left(4-\sqrt{(x-1)^2+y^2}\right)^2 \\
&\iff (x+1)^2+y^2=16-8\sqrt{(x-1)^2+y^2}+(x-1)^2+y^2 \\
&\iff (x+1)^2=16-8\sqrt{(x-1)^2+y^2}+(x-1)^2 \\
&\iff x^2+2x+1=16-8\sqrt{(x-1)^2+y^2}+x^2-2x+1 \\
&\iff 2x=16-2x-8\sqrt{(x-1)^2+y^2} \\
&\iff 4x-16=-8\sqrt{(x-1)^2+y^2} \\
&\iff x-4=-2\sqrt{(x-1)^2+y^2} \\
&\implies (x-4)^2=\left(-2\sqrt{(x-1)^2+y^2}\right)^2 \\
&\iff x^2-8x+16=4\left((x-1)^2+y^2\right) \\
&\iff x^2-8x+16=4(x^2-2x+1+y^2) \\
&\iff x^2-8x+16=4x^2-8x+4+4y^2 \\
&\iff 12=3x^2+4y^2 \\
&\iff 1=\frac{3x^2}{12}+\frac{4y^2}{12} \\
&\iff 1=\frac{x^2}{4}+\frac{y^2}{3}
\end{aligned}$$

□

17. Sketch a graph of the hyperbola $x^2 - \frac{y^2}{4} = 1$. The graph should correctly identify, the vertices, center, foci and asymptotes.

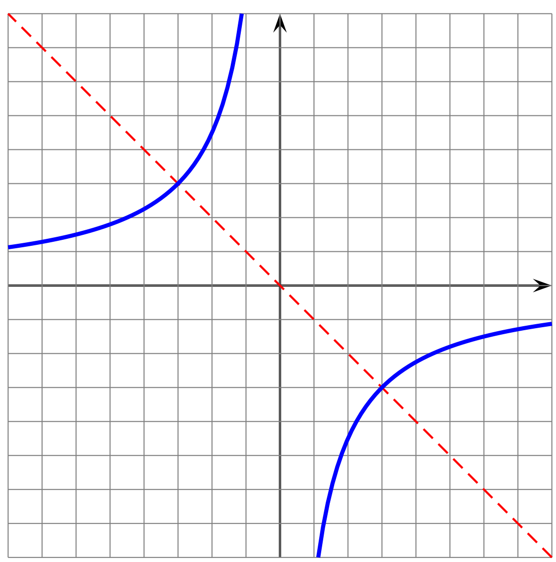
Solution. We have that $a = 1$, $b = 2$, and the hyperbola opens along the x-axis. The center is at $(0, 0)$; the vertices are at $(\pm 1, 0)$; the foci, are at $(\pm c, 0)$ where $c = \sqrt{a^2 + b^2}$; so the foci are at $(\pm\sqrt{5}, 0)$. For plotting the foci we need to know $\sqrt{5} \approx 2.2360679775$. The asymptotes have equations $y = \pm 2x$.



□

18. Sketch a graph of the hyperbola $xy = -9$. The graph should correctly identify, the vertices, center, foci and asymptotes.

Solution. This is a rectangular hyperbola with the coordinate axes as asymptotes. Since the RHS is negative the hyperbola is in the second and fourth quadrant and its axis is $y = -x$. The vertices are the points where the axis meets the hyperbola and are obtained by setting $y = -x$ in the equation of the hyperbola. We get $-x^2 = -9$ so the vertices are located at $(3, -3)$ and $(-3, 3)$. The foci are at $(9\sqrt{2}, -9\sqrt{2})$ and $(-9\sqrt{2}, 9\sqrt{2})$. We use a calculator to get $9\sqrt{2} \approx 12.7279220613$, so the foci are outside our plot window.



□

19. Find an equation of the parabola with focus at $(-3, 0)$ and directrix $x = 3$. What is the vertex and the axis of symmetry?

Solution. The parabola is the set of points whose distance from the focus equals their distance from the directrix. The square of the distance of a point with coordinates (x, y) from $(-3, 0)$ is $(x+3)^2 + y^2$, and its distance from the line $x = 3$ is $(x-3)^2$. So for a point (x, y) to be in the parabola we need

$$\begin{aligned}(x+3)^2 + y^2 = (x-3)^2 &\iff x^2 + 6x + 9 + y^2 = x^2 - 6x + 9\\ &\iff 6x + y^2 = -6x\\ &\iff y^2 = 12x\\ &\iff \frac{y^2}{12} = x\end{aligned}$$

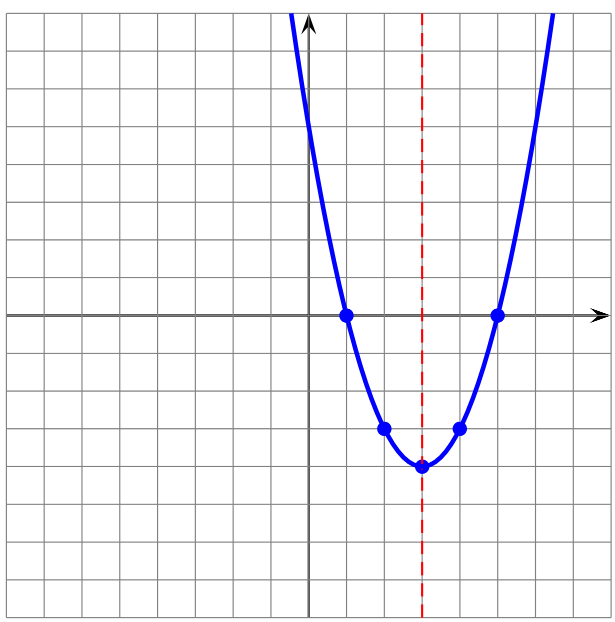
□

20. Sketch a graph of the parabola $y = x^2 - 6x + 5$. The graph should correctly indicate the vertex and the axis of symmetry.

Solution. To find the axis of symmetry and the vertex we put the equation in standard form by completing the square:

$$\begin{aligned}y = x^2 - 6x + 5 &\iff y + 9 = x^2 - 6x + 9 + 5\\ &\iff y + 9 = (x-3)^2 + 5\\ &\iff y = (x-3)^2 - 4\end{aligned}$$

So the vertex is at $(3, -4)$ and the axis of symmetry is $y = 3$. We also compute a table of values:



x	y
3	-4
2	-3
4	-3
1	0
5	0

□