Fourth Exam

Take home

Due: Thursday, April 28

1. The length of one leg of a right triangle is one unit more that the length of the other leg. The length of the hypotenuse is $\sqrt{41}$ units.

   (a) Find the lengths of the two legs.

   Solution. Let x be the length of the smaller leg, then the length of the larger leg is $x+1$. We then have by the Pythagorean theorem:

$$\begin{aligned} x^2 + (x+1)^2 = \left(\sqrt{41}\right)^2 &\iff x^2 + x^2 + 2x + 1 = 41 \\ &\iff 2x^2 + 2x - 40 = 0 \\ &\iff x^2 + x - 20 = 0 \\ &\iff (x+5)(x-4) = 0 \\ &\iff x = -5 \text{ or } x = 4 \end{aligned}$$

   Since x represents length it cannot be negative and so the solution $x = -5$ is rejected. Thus the length of the smaller leg is 4 units. It then follows that the length of the larger leg is 5 units. □

   (b) Find the measure of the two acute angles of the triangle.

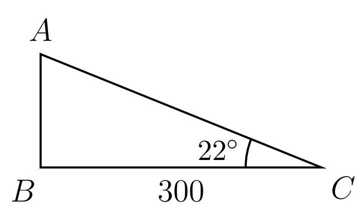
   Solution. If θ is the smaller angle of the triangle then it is opposite the smaller leg, and we have:

$$\tan\theta = \frac{4}{5} = .8$$

   Using a calculator we see that $\tan^{-1} .8 \approx 38.6598082541°$. Rounding to the nearest hundredth we have that the smaller accute angle of the triangle is approximately $38.66°$. It follows that the larger acute angle is approximately $90° - 38.66° = 51.34°$. □

2. The angle of elevation of the top of a building taken 300 feet from the base of the building is 22°. Find the height of the building to the nearest foot.

   Solution. Representing the top of the building by a point A, the point on the ground directly bellow it by B and the point where the observer stands by C we have the following figure:



If the height of the building is x feet then we have:

$$\tan 22^\circ = \frac{x}{300} \implies x = 300 \tan 22^\circ$$
$$\implies x \approx 121.207867751$$

Thus, to the nearest foot, the height of the building is 121 feet. □

3. Simplify: $5\sqrt{52} - 3\sqrt{60} + 2\sqrt{13} + 3\sqrt{135}$

   Solution. We have:

$$\begin{aligned} 5\sqrt{52} - 3\sqrt{60} + 2\sqrt{13} + 3\sqrt{135} &= 5 \cdot 2\sqrt{13} - 3 \cdot 2\sqrt{15} + 2\sqrt{13} + 3 \cdot 3\sqrt{15} \\ &= 10\sqrt{13} - 6\sqrt{15} + 2\sqrt{13} + 9\sqrt{15} \\ &= 12\sqrt{13} + 3\sqrt{15} \end{aligned}$$

□

4. Simplify: $\dfrac{2\sqrt{5} - 5\sqrt{2}}{\sqrt{10} - 2}$

   Solution. We multiply numerator and denominator with the conjugate expression of the denominator:

$$\begin{aligned} \frac{2\sqrt{5} - 5\sqrt{2}}{\sqrt{10} - 2} &= \frac{2\sqrt{5} - 5\sqrt{2}}{\sqrt{10} - 2} \cdot \frac{\sqrt{10} + 2}{\sqrt{10} + 2} \\ &= \frac{2\sqrt{50} + 4\sqrt{5} - 5\sqrt{20} - 10\sqrt{2}}{10 - 4} \\ &= \frac{2 \cdot 5\sqrt{2} + 4\sqrt{5} - 5 \cdot 2\sqrt{5} - 10\sqrt{2}}{6} \\ &= \frac{10\sqrt{2} + 4\sqrt{5} - 10\sqrt{5} - 10\sqrt{2}}{6} \\ &= \frac{-6\sqrt{5}}{6} \\ &= -\sqrt{5} \end{aligned}$$

□

5. Simplify assuming all variables represent positive numbers. The answer should contain only positive integers as exponents.

$$\left(\frac{27x^{15}y^{-\frac{21}{2}}}{8z^{-\frac{3}{2}}} \right)^{-\frac{1}{3}}$$

Solution.

$$
\begin{aligned}
\left(\frac{27x^{15}y^{-\frac{21}{2}}}{8z^{-\frac{3}{2}}}\right)^{-\frac{1}{3}} &= \frac{27^{-\frac{1}{3}}x^{-\frac{15}{3}}y^{-\frac{21}{2}\cdot\left(-\frac{1}{3}\right)}}{8^{-\frac{1}{3}}z^{-\frac{3}{2}\cdot\left(-\frac{1}{3}\right)}} \\
&= \frac{3^{-1}x^{-5}y^{\frac{7}{2}}}{2^{-1}z^{\frac{1}{2}}} \\
&= \frac{2y^3\sqrt{y}}{3x^5\sqrt{z}} \\
&= \frac{2y^3\sqrt{yz}}{3x^5z}
\end{aligned}
$$

□

6. Solve: $\sqrt{x-5}-\sqrt{x-1}=3$

Solution. We first make the LHS to contain a single radical expression and then we square both sides, then we isolate the remaining radical expression in the RHS and square again:

$$
\begin{aligned}
\sqrt{x-5}-\sqrt{x-1}=3 &\iff \sqrt{x-5}=3+\sqrt{x-1} \\
&\implies \left(\sqrt{x-5}\right)^2=\left(3+\sqrt{x-1}\right)^2 \\
&\iff x-5=9+6\sqrt{x-1}+x-1 \\
&\iff -13=6\sqrt{x-1} \\
&\implies (-13)^2=\left(6\sqrt{x-1}\right)^2 \\
&\iff 169=36(x-1) \\
&\iff 169=36x-36 \\
&\iff 205=36x \\
&\iff \frac{205}{36}=x
\end{aligned}
$$

We now need to check whether $x=\frac{205}{36}$ is a solution to the original equation:

$$
\begin{aligned}
\sqrt{\frac{205}{36}-5}-\sqrt{\frac{205}{36}-1}=3 &\iff \sqrt{\frac{205-5\cdot 36}{36}}-\sqrt{\frac{205-36}{36}}=3 \\
&\iff \sqrt{\frac{25}{36}}-\sqrt{\frac{169}{36}}=3 \\
&\iff \frac{5}{6}-\frac{13}{6}=3 \\
&\iff -\frac{8}{6}=3
\end{aligned}
$$

which is not true. Thus the original equation has no solutions. □

7. Simplify. Express your answer in the form $a + bi$ where a and b are real numbers.

$$\frac{(3+2i)^2}{12-5i}$$

Solution. We have:

$$\begin{aligned}
\frac{(3+2i)^2}{12-5i} &= \frac{(3+2i)^2}{12-5i} \\
&= \frac{9+12i-4}{12-5i} \\
&= \frac{5+12i}{12-5i} \\
&= \frac{5+12i}{12-5i} \cdot \frac{12+5i}{12+5i} \\
&= \frac{60+25i+144i-60}{144+25} \\
&= \frac{169i}{169} \\
&= i
\end{aligned}$$

□

8. Divide: $\dfrac{2x^2+4x-30}{-x^2+11x-24} \div \dfrac{-x^2+x+30}{x^2-14x+48}$. Simplify the result as much as possible.

Solution. We first invert the fraction we divide by, then we factor all numerators and denominators, simplify and perform any remaining operations:

$$\begin{aligned}
\frac{2x^2+4x-30}{-x^2+11x-24} \div \frac{-x^2+x+30}{x^2-14x+48} &= \frac{2x^2+4x-30}{-x^2+11x-24} \cdot \frac{x^2-14x+48}{-x^2+x+30} \\
&= \frac{2(x+5)(x-3)}{(8-x)(x-3)} \cdot \frac{(x-6)(x-8)}{(x+5)(6-x)} \\
&= \frac{2(x-6)(x-8)}{(6-x)(8-x)} \\
&= 2 \cdot (-1) \cdot (-1) \\
&= 2
\end{aligned}$$

□

9. Simplify: $\dfrac{\dfrac{2}{b-3} - \dfrac{3}{b+2}}{1 - \dfrac{11b+7}{b^2-b-6}}$

Solution. We multiply both numerator and denominator with the LCD $(b-3)(b+2)$:

$$\begin{aligned}
\frac{\dfrac{2}{b-3}-\dfrac{3}{b+2}}{1-\dfrac{11b+7}{b^2-b-6}} &= \frac{2(b+2)-3(b-3)}{(b-3)(b+2)-(11b+7)} \\
&= \frac{2b+4-3b+9}{b^2-b-6-11b-7} \\
&= \frac{-b+13}{b^2-12b-13} \\
&= \frac{-b+13}{(b+1)(b-13)} \\
&= \frac{-1}{b+1} \\
&= -\frac{1}{b+1}
\end{aligned}$$

□

10. Solve: $\dfrac{1}{x-2}+3=\dfrac{x}{x+3}+\dfrac{3x^2+9x-25}{x^2+x-6}$

Solution. We first find when all the expressions are defined: all denominators have to be non-zero. So we must have: $x-2\neq 0$, $x+3\neq 0$, and $x^2+x-6\neq 0$. Now, $x^2+x-6=(x+3)(x-2)$, so in order for the equation to be defined we must have $x\neq -3$ and $x\neq 2$.

Then we multiply both sides with the LCD $(x-2)(x+3)$.

$$\begin{aligned}
\frac{1}{x-2}+3=\frac{x}{x+3}+\frac{3x^2+9x-25}{x^2+x-6} &\iff x+3+3(x-2)(x+3)=x(x-2)+3x^2+9x-25 \\
&\iff x+3+3x^2+3x-18=x^2-2x+3x^2+9x-25 \\
&\iff 3x^2+4x-15=4x^2+7x-25 \\
&\iff 0=x^2-3x-10 \\
&\iff 0=(x+5)(x-2)
\end{aligned}$$

The last quadratic equation has solutions $x=-5$ and $x=2$. Since the original equation is not defined for $x=2$ that solution is rejected. Thus the only solution is $x=-5$. □

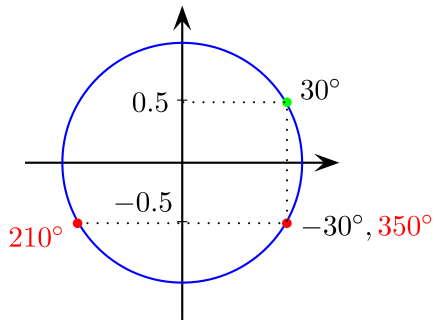
11. Find all solutions θ, with $0^\circ \leq \theta < 360^\circ$:

$$2\sin\theta = -1$$

Solution.

$$2\sin\theta=-1 \iff \sin\theta=-\frac{1}{2}$$

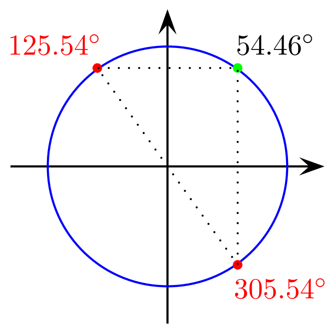
We know that $\sin 30^\circ = \dfrac{1}{2}$ so $\sin(-30^\circ) = -\dfrac{1}{2}$. Thus one solution (in IV) is $\theta = 360^\circ - 30^\circ = 350^\circ$ and the other (in III) is $180^\circ + 30^\circ = 210^\circ$.



□

12. Find all angles θ with $0° \leq \theta < 360°$ and $\tan\theta = -1.4$. Round your answers to the nearest hundredth.

Solution. Using a calculator we find, $\tan^{-1} 1.4 \approx 54.462322208$, or after rounding to the nearest hundredth, $\tan^{-1} 1.4 \approx 54.46$. So the equation $\tan\theta = -1.4$ has two solutions: one in II given by $\theta \approx 180° - 54.46° = 125.54°$ and one in IV given by $\theta \approx 180° + 125.54° = 305.54°$.



□

13. A point has coordinates $(-6, -3)$. Find its angle of reference. Round your answer to the nearest hundredth.

Solution. If θ is the angle of reference we have that θ terminates in the third quadrant and

$$\tan\theta = \frac{-6}{-3} = 2$$

Using a calculator we find $\tan^{-1} 2 \approx 63.4349488229°$ or after rounding to the near hundreth $\tan^{-1} 2 \approx 63.43°$. Since our angle lies in III we have that $\theta \approx 180° + 63.43° = 143.43°$

□

14. A point is at distance 6 from the origin and its angle of reference is 240°. Find the coordinates of the point P.

Solution. We know that if θ is the angle of reference and r is the distance of a point from the origin then

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

In our case $r = 6$ and $\theta = 240°$. So:

$$\begin{aligned} x &= 6\cos 240° &&= -3 \\ y &= 6\sin 240° &&\approx -5.1961524227 \end{aligned}$$

So rounded to the second decimal point the coordinates of P are $(-3, -5.2)$. □

15. A point P different than $(0,0)$ lies in the line with equation $y = 5x$. What are the possible angles of reference for P?

    Solution. The line $l\colon\ y = 5x$ passes through the origin $(0,0)$ and so the tangent of the angle of reference for any point in l is equal to the slope of l, that is 5. So if θ is the angle of reference for P we have that $0° \leq \theta < 360°$ and $\tan\theta = 5$.

    Using a calculator we see $\tan^{-1}\theta \approx 78.690067526°$ or, after rounding to the nearest hundredth $78.69°$. So there are two angles between $0°$ and $360°$ with $\tan\theta = 5$: the first with terminal point in the first quadrant is approximately $78.69°$ and the second with terminal point in the third quadrant is approximately $180° + 78.69° = 258.69°$. □

16. The segment with endpoints $(-1, -2)$ and $(3, 8)$ is a diameter of a circle.

    (a) Give the equation of the circle in expanded form.

    Solution. The center of the circle is the midpoint of the diameter, and its radius is half the length of the diameter. Thus the center is at

    $$\left(\frac{-1+3}{2}, \frac{-2+8}{2}\right) = (1, 3)$$

    The length of the diameter is

    $$\sqrt{(-1-3)^2 + (-2-8)^2} = \sqrt{16+100} = \sqrt{116} = 2\sqrt{29}$$

    So the radius is half of that, that is $\sqrt{29}$. Thus the equation of the circle in standard form is

    $$(x-1)^2 + (y-3)^2 = 29$$

    Expanding we get:

    $$x^2 - 2x + 1 + y^2 - 6y + 9 = 29$$

    and finally after simplifying:

    $$x^2 - 2x + y^2 - 6y = 19$$

    □

(b) What's the length of the circumference of this circle?

Solution. The length of the circumference is π times the length of the diameter. In part (a) of the exercise we found that the diameter has length $2\sqrt{29}$. Thus the length of the circumpherance of this circle is $2\pi\sqrt{29}$ units. □

17. Find the common points of the circle $x^2 - 6x + y^2 + 4y = 87$ and the line $y = 3 - x$.

Solution. We have to solve the system:

$$\begin{cases} x^2 - 6x + y^2 + 4y &= 87 \\ y &= 3 - x \end{cases}$$

We substitute the second equation into the first, expand, simplify and solve:

$$\begin{aligned} x^2 - 6x + (3-x)^2 + 4(3-x) = 87 &\iff x^2 - 6x + 9 - 6x + x^2 + 12 - 4x = 87 \\ &\iff 2x^2 - 16x + 21 = 87 \\ &\iff 2x^2 - 16x - 66 = 0 \\ &\iff x^2 - 8x - 33 = 0 \end{aligned}$$

This is a quadratic equation. It's discriminant is $D = (-8)^2 - 4 \cdot 1 \cdot (-33) = 196 = 14^2$ and so it has two solutions:

$$x = \frac{8 \pm 14}{2} = \begin{cases} 11 \\ -3 \end{cases}$$

Substituting $x = 11$ in the second equation of the system gives $y = -8$ while substituting $x = -3$ gives $y = 6$. Thus the circle and the line intersect at the points $(11, -8)$ and $(-3, 6)$. □

18. Consider the circle C with equation $x^2 + 2x + y^2 - 6y = 0$.

(a) Verify that the point $P(2, 4)$ lies on the circle C.

Solution. We just substitute the coordinates of the point in to the equation of the circle:

$$2^2 + 2 \cdot 2 + 4^2 - 6 \cdot 4 = 0 \iff 4 + 4 + 16 - 24 = 0$$

which is a true equation. Thus the point lies on the circle. □

(b) Find an equation for the line tangent to C at P.

Solution. We first find the center of the circle by writing the equation of the circle in standard form:

$$\begin{aligned} x^2 + 2x + y^2 - 6y = 0 &\iff x^2 + 2x + 2 + y^2 - 6y + 9 = 0 + 1 + 9 \\ &\iff (x+1)^2 + (y-3)^2 = 10 \end{aligned}$$

Thus the center of the circle is at $(-1, 3)$. So the radius to P has slope $\dfrac{4-3}{2+1} = \dfrac{1}{3}$ and therefore the slope of the tangent line is -3. So the equation of the tangent line is

$$y - 4 = -3(x-2) \iff y - 4 = -3x + 6 \iff y = -3x + 10$$

$\square$

19. Find the standard form of the equation of the ellipse with foci at $(0, -3)$ and $(0, 3)$, given that the sum of the distances of a point in the ellipse from the two foci is 10.

Solution. For a point (x, y) in the ellipse we have that the sum of distances from the two foci is 10. So we have:

$$\begin{aligned}
&\sqrt{x^2+(y+3)^2} + \sqrt{x^2+(y-3)^2} = 10 \\
\iff &\sqrt{x^2+(y+3)^2} = 10 - \sqrt{x^2+(y-3)^2} \\
\implies &\left(\sqrt{x^2+(y+3)^2}\right)^2 = \left(10 - \sqrt{x^2+(y-3)^2}\right)^2 \\
\iff &x^2 + (y+3)^2 = 100 - 20\sqrt{x^2+(y-3)^2} + x^2 + (y-3)^2 \\
\iff &y^2 + 6y + 9 = 100 - 20\sqrt{x^2+(y-3)^2} + y^2 - 6y + 9 \\
\iff &6y = 100 - 20\sqrt{x^2+(y-3)^2} - 6y \\
\iff &12y - 100 = -20\sqrt{x^2+(y-3)^2} \\
\iff &3y - 25 = -5\sqrt{x^2+(y-3)^2} \\
\implies &(3y-25)^2 = \left(-5\sqrt{x^2+(y-3)^2}\right)^2 \\
\iff &9y^2 - 150y + 625 = 25\left(x^2 + (y-3)^2\right) \\
\iff &9y^2 - 150y + 625 = 25\left(x^2 + y^2 - 6y + 9\right) \\
\iff &9y^2 - 150y + 625 = 25x^2 + 25y^2 - 150y + 225 \\
\iff &400 = 25x^2 + 16y^2 \\
\iff &1 = \frac{x^2}{16} + \frac{y^2}{25}
\end{aligned}$$

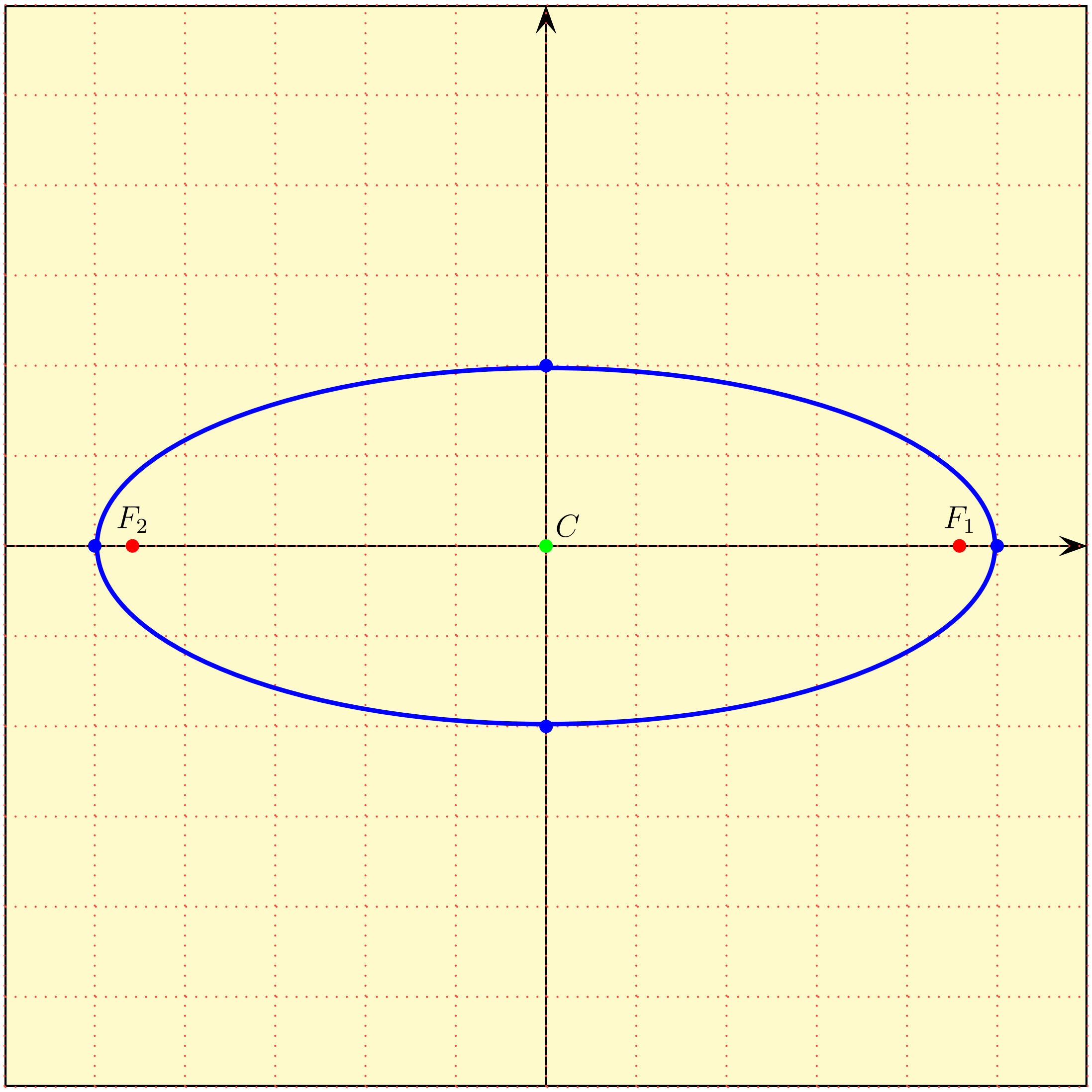
$\square$

20. Sketch the graph of the ellipse:

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

The graph should correctly reflect the minor and major axis, the center and the foci.

Solution. The center of the ellipse is at $(0, 0)$. The major axis is along the x-axis and has length $2\sqrt{25} = 10$. The minor axis is along the y-axis and has length $2\sqrt{4} = 4$. The coordinates of the foci are $(\pm c, 0)$ where $c = \sqrt{25-4} = \sqrt{21}$. For graphing the foci we need $\sqrt{21} \approx 4.58257569496$. So we have the following graph:



F_2
C
F_1

□