Third Exam

Take home

The answers

1. One leg of a right triangle is 2 cm more than the other. If the hypotenuse is $\sqrt{7}$ cm

   (a) find the lengths of the two legs.

   Solution. Let x be the length of the smaller leg, in centimeters, then the larger leg has length $x+2$. The Pythagorean theorem then gives:

   $$\begin{aligned} x^2+(x+2)^2=\left(\sqrt{7}\right)^2 &\iff x^2+x^2+4x+4=7 \\ &\iff 2x^2+4x-3=0 \end{aligned}$$

   We solve the last equation using the quadratic formula. The discriminant is $D = 4^2 - 4\cdot 2\cdot(-3) = 40$ so the two solutions are:

   $$x = \frac{-4\pm\sqrt{40}}{4} = \frac{-4\pm 2\sqrt{10}}{4} = \frac{-2\pm\sqrt{10}}{2}$$

   Since lengths of sides are positive, the solution $x=\frac{-2-\sqrt{10}}{2}$ is rejected. So the length of the smaller leg is $\frac{-2+\sqrt{10}}{2}$. The length of the larger leg is then

   $$\frac{-2+\sqrt{10}}{2}+2=\frac{2+\sqrt{10}}{2}$$

   □

   (b) Find the measures of the two acute angles of the triangle.

   Solution. Let A, B, and C be the angles of the triangle with $A=90°$, B the acute angle opposite of the smaller leg and C the acute angle opposite of the larger leg. Then we can compute the tangent of the angle B:

   $$\begin{aligned} \tan B &= \frac{\frac{-2+\sqrt{10}}{2}}{\frac{2+\sqrt{10}}{2}} \\ &= \frac{-2+\sqrt{10}}{2+\sqrt{10}} \\ &\approx 0.225148226555 \end{aligned}$$

   So $B\approx\tan^{-1}(0.225148226555)=12.6884667626°$. Rounding to the nearest hundredth we have $B\approx 12.69°$. Then $C\approx 90°-12.69° = 77.31°$. □

2. Simplify: $3\sqrt{28}-\sqrt{700}+4\sqrt{63}$

Solution.

$$
\begin{aligned}
3\sqrt{28} - \sqrt{700} + 4\sqrt{63} &= 3\sqrt{4 \cdot 7} - \sqrt{7 \cdot 100} + 4\sqrt{7 \cdot 9} \\
&= 3 \cdot 2\sqrt{7} - 10\sqrt{7} + 4 \cdot 3\sqrt{7} \\
&= 6\sqrt{7} - 10\sqrt{7} + 12\sqrt{7} \\
&= 8\sqrt{7}
\end{aligned}
$$

□

3. Simplify: $\dfrac{\left(3-\sqrt{2}\right)^2}{1+\sqrt{2}}$

Solution.

$$
\begin{aligned}
\frac{\left(3-\sqrt{2}\right)^2}{1+\sqrt{2}} &= \frac{9 - 6\sqrt{2} + 2}{1+\sqrt{2}} \\
&= \frac{11 - 6\sqrt{2}}{1+\sqrt{2}} \\
&= \frac{11 - 6\sqrt{2}}{1+\sqrt{2}} \cdot \frac{1-\sqrt{2}}{1-\sqrt{2}} \\
&= \frac{11 - 11\sqrt{2} - 6\sqrt{2} + 12}{1-2} \\
&= \frac{23 - 17\sqrt{2}}{-1} \\
&= 17\sqrt{2} - 23
\end{aligned}
$$

□

4. Simplify, assuming all variables represent positive numbers: $\sqrt{\dfrac{9b^8c^3}{20a^7}}$

Solution.

$$
\begin{aligned}
\sqrt{\frac{9b^8c^3}{20a^7}} &= \frac{3b^4c\sqrt{c}}{2\sqrt{5}a^3\sqrt{a}} \\
&= \frac{3b^4c\sqrt{c}}{2a^3\sqrt{5a}} \cdot \frac{\sqrt{5a}}{\sqrt{5a}} \\
&= \frac{3b^4c\sqrt{5ac}}{10a^4}
\end{aligned}
$$

□

5. Simplify assuming all variables represent positive numbers. The answer should contain only positive integers as exponents.

$$\left(\frac{x^{21}y^{-\frac{15}{4}}}{z^{-\frac{9}{2}}}\right)^{-\frac{2}{3}}$$

Solution.

$$\begin{aligned}\left(\frac{x^{21}y^{-\frac{15}{4}}}{z^{-\frac{9}{2}}}\right)^{-\frac{2}{3}} &= \frac{x^{21\frac{-2}{3}}y^{-\frac{15}{4}\frac{-2}{3}}}{z^{-\frac{9}{2}\frac{-2}{3}}} \\ &= \frac{x^{-14}y^{\frac{5}{2}}}{z^{3}} \\ &= \frac{y^2\sqrt{y}}{x^{14}z^3}\end{aligned}$$

□

6. Solve: $x - \sqrt{x-4} = 10$

Solution. We first isolate the radical, and then square both sides:

$$\begin{aligned}x - \sqrt{x-4} = 10 &\iff -\sqrt{x-4} = 10 - x \\ &\implies \left(-\sqrt{x-4}\right)^2 = (10-x)^2 \\ &\iff x - 4 = 100 - 20x + x^2 \\ &\iff 0 = 104 - 21x + x^2\end{aligned}$$

The last equation is quadratic. We solve using the quadratic formula: the discriminant is $D = (-21)^2 - 4 \cdot 1 \cdot 104 = 25 = 5^2$. So we have two solutions:

$$x = \frac{21 \pm 5}{2} = \begin{cases} 13 \\ 8 \end{cases}$$

We now need to check whether these are solutions of the original equation. We first check $x = 13$:

$$13 - \sqrt{13-4} = 10 \iff 13 - 3 = 10$$

which is a true equation. Thus $x = 13$ is a solution.

We then check $x = 8$:

$$8 - \sqrt{8-4} = 10 \iff 8 - 2 = 10$$

which is a false equation. Thus $x = 8$ is extraneous. So, the original equation has only one solution $x = 13$. □

7. Solve: $\sqrt{x} - \sqrt{x-9} = 1$

Solution. We first make the LHS to contain a single radical expression and then we square both sides, then we isolate the remaining radical expression in the RHS and square again:

$$\begin{aligned}\sqrt{x}-\sqrt{x-9}=1 &\iff -\sqrt{x-9}=1-\sqrt{x}\\ &\implies \left(-\sqrt{x-9}\right)^2=\left(1-\sqrt{x}\right)^2\\ &\iff x-9=1-2\sqrt{x}+x\\ &\iff -10=-2\sqrt{x}\\ &\iff 5=\sqrt{x}\\ &\implies (5)^2=\left(\sqrt{x}\right)^2\\ &\iff 25=x\end{aligned}$$

We now need to check whether $x=25$ is a solution to the original equation:

$$\sqrt{25}-\sqrt{25-9}=1 \iff 5-4=1$$

which is a true equation. Thus $x=25$ is the solution. □

8. Simplify. Express your answer in the form $a+bi$ where a and b are real numbers.

$$\frac{(2-3i)(i+1)+2+12i}{3-5i}$$

Solution.

$$\begin{aligned}\frac{(2-3i)(i+1)+2+12i}{3-5i} &= \frac{2i+2+3-3i+2+12i}{3-5i}\\ &= \frac{7+11i}{3-5i}\\ &= \frac{7+11i}{3-5i}\cdot\frac{3+5i}{3+5i}\\ &= \frac{21+35i+33i-55}{9+25}\\ &= \frac{-34+88i}{34}\\ &= -1+2i\end{aligned}$$

□

9. Simplify: $\dfrac{x^2+10x+25}{x^2+2x-15}$

Solution. We factor numerator and denominator and then cancel possible common factors:

$$\begin{aligned}\frac{x^2+10x+25}{x^2+2x-15} &= \frac{(x+5)^2}{(x+5)(x-3)}\\ &= \frac{x+5}{x-3}\end{aligned}$$

$\square$

10. Divide : $\dfrac{x^2-4}{x^2+x-6} \div \dfrac{x^2+7x+10}{x^2+8x+15}$. Simplify the result as much as possible.

Solution.

$$\begin{aligned}
\frac{x^2-4}{x^2+x-6} \div \frac{x^2+7x+10}{x^2+8x+15} &= \frac{x^2-4}{x^2+x-6} \div \frac{x^2+7x+10}{x^2+8x+15} \\
&= \frac{(x+2)(x-2)}{(x+3)(x-2)} \cdot \frac{(x+3)(x+5)}{(x+2)(x+5)} \\
&= \frac{x+2}{x+3} \cdot \frac{x+3}{x+2} \\
&= 1
\end{aligned}$$

$\square$

11. Combine: $\dfrac{2}{x} - \dfrac{2x-3}{x^2-25} + \dfrac{5}{x-5}$. Simplify the result as much as possible.

Solution. The LCD of all denominators is $x(x-5)(x+5)$. So we have:

$$\begin{aligned}
\frac{2}{x} - \frac{2x-3}{x^2-25} + \frac{5}{x-5} &= \frac{2(x-5)(x+5)}{x(x-5)(x+5)} - \frac{x(2x-3)}{x(x^2-25)} + \frac{5x(x+5)}{x(x-5)(x+5)} \\
&= \frac{2x^2-50-2x^2+3x+5x^2+25x}{x(x-5)(x+5)} \\
&= \frac{5x^2+28x-50}{x(x-5)(x+5)}
\end{aligned}$$

$\square$

12. Simplify: $\dfrac{\dfrac{a}{a-3} - \dfrac{3}{a+3}}{1 + \dfrac{18}{a^2-9}}$

Solution. We multiply both numerator and denominator with the LCD a^2-9:

$$\begin{aligned}
\frac{\dfrac{a}{a-3} - \dfrac{3}{a+3}}{1 + \dfrac{18}{a^2-9}} &= \frac{a(a+3)-3(a-3)}{a^2-9+18} \\
&= \frac{a^2+3a-3a+9}{a^2+9} \\
&= \frac{a^2+9}{a^2+9} \\
&= 1
\end{aligned}$$

$\square$

13. Solve: $\dfrac{1}{x^2} - 15 = -\dfrac{2}{x}$

Solution. We first find when all the expressions are defined: all denominators have to be non-zero. So we must have: $x \neq 0$. Then we multiply both sides with the LCD x^2.

$$\begin{aligned} \frac{1}{x^2} - 15 = -\frac{2}{x} &\iff 1 - 15x^2 = -2x \\ &\iff 0 = 15x^2 - 2x - 1 \end{aligned}$$

The last equation is quadratic. Its discriminant is $D = (-2)^2 - 4 \cdot 15 \cdot (-1) = 64 = 8^2$. So the solutions are:

$$x = \frac{2 \pm 8}{30} = \begin{cases} \dfrac{1}{3} \\ -\dfrac{1}{5} \end{cases}$$

The equation is defined for both solutions, so they are both accepted. □

14. Solve: $\dfrac{2}{x+7} + 2 = \dfrac{1}{x-3} - \dfrac{4x+48}{x^2+4x-21}$

Solution. The equation is defined for $x \neq -7$ and $x \neq 3$. The LCD of all denominators is $(x+7)(x-3)$, so after multiplying both sides with the LCD we obtain:

$$\begin{aligned} \frac{2}{x+7} + 2 = \frac{1}{x-3} - \frac{4x+48}{x^2+4x-21} &\iff 2(x-3) + 2(x-3)(x+7) = x + 7 - (4x+48) \\ &\iff 2x - 6 + 2x^2 + 8x - 42 = x + 7 - 4x - 48 \\ &\iff 2x^2 + 10x - 48 = -3x - 41 \\ &\iff 2x^2 + 13x - 7 = 0 \end{aligned}$$

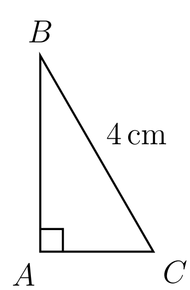
The discriminant of the last equation is $13^2 - 4 \cdot 2 \cdot (-7) = 225 = 15^2$. So we get the solutions;

$$x = \frac{-13 \pm 15}{4} = \begin{cases} \dfrac{1}{2} \\ -7 \end{cases}$$

The equation is not defined for $x = -7$ so this solution is rejected. Thus the only solution is $x = \dfrac{1}{2}$ □

15. Solve the triangle ABC, using the given information:

$A = 90°$	$a = 4$ cm
$B = 30°$	$b =$
$C =$	$c =$



Solution. The sum of the two acute angles of the triangle is $90°$. So, $B + C = 90° \implies 30° + C = 90° \implies C = 60°$.

Since $B = 30°$ we have:

$$\sin 30° = \frac{b}{4} \iff \frac{1}{2} = \frac{b}{4}$$
$$\iff b = 2$$

To find C we can use the Pythagorean theorem:

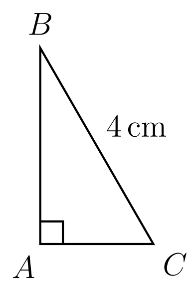
$$c^2 + 2^2 = 4^2 \iff c^2 + 4 = 16$$
$$\iff c^2 = 12$$
$$\iff c = \sqrt{12}$$
$$\iff c = 2\sqrt{3}$$

In sum:

$A = 90°$ $\quad a = 4$ cm

$B = 30°$ $\quad b = 2$ cm

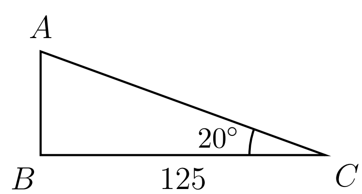
$C = 60°$ $\quad c = 2\sqrt{3}$ cm



□

16. A hot-air balloon rises vertically. An observer stands on level ground at a distance of 125 feet from a point on the ground directly below the passenger's compartment. How high, to the nearest foot, is the balloon if the angle of elevation is $20°$?

Solution. Representing the passenger compartment of the balloon by a point A, the point on the ground directly bellow it by B and the point where the observer stands by C we have the following figure:



If the height of the balloon is x feet then we have:

$$\tan 20° = \frac{x}{125} \implies x = 125 \tan 20°$$
$$\implies x \approx 45.4962792833$$

Thus, to the nearest foot, the balloon is 45 feet high. □

17. An angle θ has $\tan\theta = 1.1917536$.

    (a) Based on this information in which quadrants can the terminal point of θ lie?

    Solution. Since the tangent of the angle is positive its terminal point lies either in quadrant I or quadrant III. $\square$

    (b) Find all possible such angles θ, with $0° \leq \theta < 360°$.

    Solution. Using a calculator we see that $\tan^{-1} 1.1917536 \approx 50°$. So one of the angles (the one in the first quadrant) is $50°$. The other angle (the one in the third quadrant) is $180° + 50° = 230°$. $\square$

18. A point has coordinates $(-2, 5)$. Find its angle of reference.

    Solution. If θ is the angle of reference then the terminal point of θ is in quadrant II and we have:

$$\begin{aligned} \tan\theta = \frac{y}{x} &\Longrightarrow \tan\theta = \frac{5}{-2} \\ &\Longrightarrow \tan\theta = -2.5 \end{aligned}$$

    Using the calculator we see that $\tan^{-1} 2.5 \approx 68.2°$. So the angle in quadrant II with $\tan\theta = -2.5$ is approximately

$$\theta \approx 180° - 68.2 = 111.8$$

$\square$

19. A point is at distance 7 from the origin and has angle of reference $140°$. Find its coordinates.

    Solution. We know that if θ is the angle of reference and r is the distance of a point from the origin then

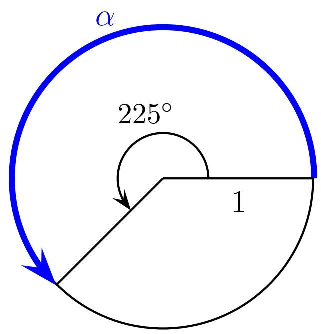
$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

    In our case $r = 7$ and $\theta = 140°$. So:

$$\begin{aligned} x &= 7\cos 140° &\approx -5.36231110183 \\ y &= 7\sin 140° &\approx 4.49951326781 \end{aligned}$$

    So rounded to the second decimal point the coordinates of P are $(-5.36, 4.50)$. $\square$

20. Find the length of the arc α, where the corner of the angle is at the center of the circle. Give an exact answer.



Solution. The arc α is $\frac{225}{360} = \frac{5}{8}$ of the whole circle. So its length will be $\frac{5}{8}$ of the length of the whole circle. Now since the radius is 1 the length of the whole circle is 2π. It follows that the length of the arc α is:

$$\frac{5}{8}(2\pi) = \frac{5\pi}{4}$$

□