Exercises for "graphs of polynomials"

(1) For each of the following, sketch a rough graph of the given function. The graph should correctly reflect the end behavior, the behavior near zeros and the number of turning points. The y–intercept should also be correctly marked.
  (a) $p(x) = x^4 + 4x^3 + 6x^2 + 4x + 1$
  (b) $g(x) = x^3 - 6x^2 + 12x - 8$
  (c) $h(x) = 6x^3 - x^2 - 11x + 6$
  (d) $k(x) = x^4 - 11x^2 + 24$
  (e) $f(x) = -2x^4 + 4x^3 + 22x^2 - 24x - 72$
  (f) $f(x) = x^5 + 2x^4 - 6x^3 - 8x^2 + 5x + 6$
  (g) $g(x) = x^4 - 5x^3 + x^2 + 21x - 18$
  (h) $q(x) = 3(2 - x)(x - 3)(x + 1)^2$
  (i) $p(x) = -2x^2(x - 3)^5(x + 2)^2(x - 1)(3x - 4)^3$
  (j) $f(x) = (-2x + 1)(2x + 3)(x - 4)^3(1 - x)^2$

(2) Solve the following inequalities: (you may use the results from the previous exercise).
  (a) $x^5 + 2x^4 - 6x^3 - 8x^2 + 5x + 6 \leq 0$
  (b) $x^3 - 6x^2 + 12x - 8 \geq 0$
  (c) $x^4 - 5x^3 + x^2 + 21x - 18 < 0$
  (d) $-2x^4 + 4x^3 + 22x^2 - 24x - 72 > 0$
  (e) $x^4 + 4x^3 + 6x^2 + 4x + 1 \leq 0$

(3) For each of the following lists of properties, give an example of a polynomial $p(x)$ that has all of the properties.
  (a) The degree of $p(x)$ is 3 and its graph intercepts the x–axis at the points $x = 0$, $x = 1$ and $x = 3$. Additionally as $x \to \infty$, $p(x) \to -\infty$.
  (b) The degree of $p(x)$ is 3. The zeros of $p(x)$ are $-1, 2, 3$ and its constant term is 12.
  (c) The only x-intercepts of $y = p(x)$ are $x = -3$, $x = 1$, and $x = 2$. As $x \to \infty$, $p(x) \to \infty$ and as $x \to -\infty$, $p(x) \to \infty$. The y–intercept of $y = p(x)$ is at $y = 18$.
  (d) The solution set of the inequality $p(x) < 0$ is empty and the polynomial has exactly two real roots $x = 1$ and $x = -1$. Additionally the leading coefficient is 4 and the constant term is 8.

(4) Give two different proofs of the following: A real polynomial of odd degree has at least one real root.

(5) **Extra Credit:** Consider the following function:

$$f(x) = 3x^4 + 4x^3 - 13x^2 + 12x - 4$$

Prove that this function has at least one zero in the interval $(0, 1)$, i.e. prove that for some a with $0 < a < 1$ we have that $f(a) = 0$.