BRONX COMMUNITY COLLEGE
of the City University of New York

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

MATH 30 — Midterm
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Name: KEY.

Directions: Write your answers in the provided space. To get full credit you *must* show all your work. Simplify your answers whenever possible. Be certain to indicate your final answer clearly. This exam has a total of 100 points.

1. Find the domain for each of the following functions:

(a) (10 points) $f(x) = \dfrac{2x-1}{x^2-x-12}$ We need $x^2-x-12 \neq 0$

Now $x^2-x-12 = (x-4)(x+3)$.
So we need $x \neq 4$ and $x \neq -3$. $-\infty \quad -3 \quad 4 \quad +\infty$

Thus the domain is $(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$.

(b) (10 points) $g(x) = \sqrt{x^2-4}$ We need $x^2-4 \geq 0$. Now $x^2-4 = (x-2)(x+2)$. We have the following table of signs:

So domain is $(-\infty, -2] \cup [2, \infty)$

	$-\infty$	-2		2	$+\infty$
$x+2$	$-$	0	$+$		$+$
$x-2$	$-$		$-$	0	$+$
	$+$	0	$-$	0	$+$

(c) (10 points) $h(x) = \log_2(9-x^2)$

We need $9-x^2 > 0$. Now $9-x^2 = (3+x)(3-x)$. So we have the following table:

So domain is: $(-3, 3)$.

	$-\infty$	-3		3	$+\infty$
$3+x$	$-$	0	$+$		$+$
$3-x$	$+$		$+$	0	$-$
	$-$	0	$+$	0	$-$

2. (10 points) Find $f \circ g$, where $f(x) = \dfrac{2x-3}{5x+2}$ and $g(x) = \dfrac{x-1}{x+2}$

First the formula:

$$(f \circ g)(x) = f(g(x)) = \frac{2\left(\frac{x-1}{x+2}\right) - 3}{5\left(\frac{x-1}{x+2}\right) + 2} \cdot \frac{x+2}{x+2}$$

$$= \frac{2(x-1) - 3(x+2)}{5(x-1) + 2(x+2)}$$

$$= \frac{2x - 2 - 3x - 6}{5x - 5 + 2x + 4}$$

$$= \frac{-x-8}{7x-1} = \boxed{-\frac{x+8}{7x-1}}$$

Now the domain.

x is in the domain of $f \circ g$, when both of the following two conditions hold:

1) x is in domain of $g \iff x+2 \neq 0 \Rightarrow x \neq -2$

2) g(x) is in domain of $f \iff 7x-1 \neq 0 \Rightarrow x \neq \frac{1}{7}$.

Combining these two conditions we have:

domain is

$(-\infty, -2) \cup (-2, 1/7) \cup (1/7, \infty)$.

3. (10 points) Let $f(x) = x^2 - 4x - 2$ with domain $(-\infty, 2]$, and $g(x) = 2 - \sqrt{x+6}$. Prove that f and g are a pair of inverse functions.

We will show that a) for x in domain of g, $f(g(x)) = x$
b) for x in domain of f, $g(f(x)) = x$.

a) $f(g(x)) = (2 - \sqrt{x+6})^2 - 4(2 - \sqrt{x+6}) - 2$
$= \cancel{4} - \cancel{4\sqrt{x+6}} + x + 6 - \cancel{8} + \cancel{4\sqrt{x+6}} - \cancel{2}$
$= x.$

b) $g(f(x)) = 2 - \sqrt{(x^2-4x-2)+6}$
$= 2 - \sqrt{x^2-4x+4}$
$= 2 - \sqrt{(x-2)^2}$
$= 2 - |x-2|$

Now if x is in the domain of f, $x - 2 < 0$. So $|x-2| = -x+2$.
So $g(f(x)) = 2 - (-x+2)$
$= x$

4. (10 points) Find the formula, the domain and the range of f^{-1}, where

$$f(x) = \frac{-x+3}{4x-7}$$

f as a relation is $y = \frac{-x+3}{4x-7}$. So f^{-1} is $x = \frac{-y+3}{4y-7}$

We solve for y: $x = \frac{-y+3}{4y-7} \iff x(4y-7) = -y+3$
$\iff 4xy - 7x = -y + 3$
$\iff 4xy + y = 7x + 3$
$\iff (4x+1)y = 7x+3$
$\iff y = \frac{7x+3}{4x+1}.$

Thus the formula is
$f^{-1}(x) = \frac{7x+3}{4x+1}$.

Domain of f^{-1}: we need $4x+1 \neq 0$
So $x \neq -\frac{1}{4}$. Thus domain is $(-\infty, -\frac{1}{4}) \cup (-\frac{1}{4}, \infty)$

Range of f^{-1} is the domain of f.
$4x - 7 \neq 0 \iff x \neq \frac{7}{4}$
So ~~domain~~ range is $(-\infty, \frac{7}{4}) \cup (\frac{7}{4}, \infty)$.

5. (5 points) List all possible rational roots of the following polynomial, according to the "Rational Root Theorem".

$$p(x) = 6x^5 - 3x^4 + 7x^3 - 2x^2 + 8x - 12$$

Possible numerators: $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$

Possible denominators: $\pm1, \pm2, \pm3, \pm6$

So possible solutions (after eliminating repetitions):

$\pm1, \pm2, \pm3, \pm4, \pm6, \pm12, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{1}{6}$

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6. (15 points) Solve the following equation:

$$x^5 - 5x^4 - x^3 + 11x^2 - 6 = 0$$

Possible rational roots: $\pm1, \pm2, \pm3, \pm6$.

We try them using synthetic division:

$$\begin{array}{r|rrrrrr}
1 & 1 & -5 & -1 & 11 & 0 & -6 \\
 & & 1 & -4 & -5 & 6 & 6 \\
\hline
1 & 1 & -4 & -5 & 6 & 6 & \boxed{0} \checkmark \\
 & & 1 & -3 & -8 & -2 & \\
\hline
 & 1 & -3 & -8 & -2 & \boxed{4} \times &
\end{array}$$

$$\begin{array}{r|rrrrr}
-1 & 1 & -4 & -5 & 6 & 6 \\
 & & -1 & 5 & 0 & -6 \\
\hline
-1 & 1 & -5 & 0 & 6 & \boxed{0} \checkmark \\
 & & -1 & 6 & -6 & \\
\hline
 & 1 & -6 & 6 & \boxed{0} \checkmark &
\end{array}$$

So the L.H.S. factors as:

$(x-1)(x+1)^2(x^2-6x+6)$

So we have

$\boxed{x=1}$ or $\boxed{x=-1}$ (double sol.)

or $x^2 - 6x + 6 = 0$

$D = 36 - 24$
$= 12.$

Solutions: $x = \dfrac{6 \pm \sqrt{12}}{2}$

$= \dfrac{6 \pm 2\sqrt{3}}{2}$

$= \boxed{3 \pm \sqrt{3}.}$

In sum, $x = 1$, or $x = -1$,
or $x = 3 + \sqrt{3}$
or $x = 3 - \sqrt{3}$

7. (20 points) Solve the following inequality:

$$x^4 - 4x^3 + 3x^2 + 4x - 4 \geq 0$$

First we factor. We check possible rational roots:

± 1, ± 2, and ± 4

$$\begin{array}{r|rrrrr}
1 & 1 & -4 & 3 & 4 & -4 \\
 & & 1 & -3 & 0 & 4 \\
\hline
1 & 1 & -3 & 0 & 4 & \boxed{0} \checkmark \\
 & & 1 & -2 & -2 & \\
\hline
 & 1 & -2 & -2 & 2 \times &
\end{array}$$

$$\begin{array}{r|rrrr}
-1 & 1 & -3 & 0 & 4 \\
 & & -1 & 4 & \cancel{0}-4 \\
\hline
 & 1 & -4 & 4 & \boxed{0} \checkmark
\end{array}$$

So $x^4 - 4x^3 + 3x^2 + 4x - 4$

$= (x-1)(x+1)(x^2 - 4x + 4)$

$= (x-1)(x+1)(x-2)^2$

	$-\infty$	-1		1		2	$+\infty$
$x+1$	$-$	0	$+$		$+$		$+$
$x-1$	$-$		$-$	0	$+$		$+$
$(x+2)^2$	$+$		$+$		$+$	0	$+$
	$+$	0	$-$	0	$+$	0	$+$

So solution is $\boxed{(-\infty, -1] \cup [1, \infty)}$