Second Quiz for CSI35

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Directions: This quiz is due Thursday March 5, at 6:00 PM.

1. For an integer n let g_n be the number of ways that n can be written as a sum of ones and twos, where the order that the summands are written is important. For example, $g(1) = 1$, $g(2) = 2$ since 2 can be written either as 2 or as $1 + 1$, and $g(3) = 3$ because 3 can be written as $1 + 1 + 1$ or as $1 + 2$ or as $2 + 1$.
   (a) Find a recursive definition of $g(n)$
   (b) Prove that this recursive definition is correct.

   For the next three questions f_n stands for the n th Fibonacci number.

2. Prove that for all $n \geq 1$ we have:

$$f_1^2 + f_2^2 + \cdots f_n^2 = f_n f_{n+1}$$

3. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Prove that for all $n \geq 1$, $A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$.

4. Let $\varphi = \dfrac{1 + \sqrt{5}}{2}$, $\bar{\varphi} = \dfrac{1 - \sqrt{5}}{2}$.
   (a) Prove that $\forall n \in \mathbb{N} \quad \varphi^n = f_{n-1} + f_n \varphi$ and $\bar{\varphi}^n = f_{n-1} + f_n \bar{\varphi}$.
   (b) Prove that

$$\forall n \in \mathbb{N} \quad f_n = \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}}$$

   For the next four questions recall that if $\Sigma = \{0, 1\}$ then the elements of Σ^*, i.e. the words on the alphabet Σ, are called *bit strings.*

5. How many bit strings of length n are there, where n is any natural number? Prove your answer.

6. For a bit string s, let $O(s)$ and $I(s)$ be number of zeroes and ones, respectively, that occur in s. So for example if $s = 01001$, then $O(s) = 3$ and $I(s) = 2$.

(a) Give recursive definitions of $O(s)$ and $I(s)$.

(b) If $l(s)$ stands for the length of s, prove that:

$$l(s) = O(s) + I(s)$$

7. The reverse of a string s is the string obtained by "reading s backwards", for example the reverse of the string "sub" is "bus". The reverse of a string s is denoted by s^R. Give a recursive definition of s^R, for bit strings s.

8. A *palindrome* is a string s such that $s^R = s$, in other words a string that reads the same when we read it backwards. For example the string "bob" is a palindrome.

   (a) Give a recursive definition of the set Π of all bit strings that are palindromes.

   (b) For a natural number n, how many bit string palindromes of length n are there? Prove your answer.

9. The set of *binary trees*, is recursively defined as follows:

   - There is a a binary tree consisting of a single vertex r. The root of this tree is r.
   - If T_1 and T_2 are two binary trees with roots r_1 and r_2 respectively, we can make a new binary tree by adding one new vertex r and two new edges connecting r to r_1 and r_2. The root of this new tree is r.
   - All binary trees are constructed this way

   For a binary tree, T let $v(T)$ and $e(T)$ denote the number of vertices and edges of T respectively.

   (a) Give recursive definitions of $v(T)$ and $e(T)$.

   (b) Prove that for all binary trees:

$$v(T) - e(T) = 1$$

10. **Extra Credit:** Give a recursive definition for $R(n)$ the number of rooted trees with n vertices.