First Quiz for CSI35

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Directions: This quiz is due Tuesday February 10, at 6:00 PM.

1. Prove that for all natural numbers $n \geq 1$ we have:
$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

2. Prove that for all $n \in \mathbb{N}$, 5 divides $n^5 - n$.

3. Prove that for all $n \in \mathbb{N}$, 6 divides $n^3 - n$.

4. Alice and Bob play a game by taking turns removing 1, 2 or 3 stones from a pile that initially has n stones. The person that removes the last stone wins the game. Alice plays always first.
   (a) Prove by induction that if n is a multiple of 4 then Bob has a wining strategy.
   (b) Prove that if n is not a multiple of 4 then Alice has a wining strategy.

5. Chris and Dominique play a slightly different game. Again each player takes turns removing 1, 2 or 3 stones from a pile that initially has n stones but now, the person that removes the last stone loses the game. Chris plays always first. Analyze this game, that is, find the values of n for which Chris has a winning strategy and the values of n for which Dominique has a winning strategy. You should prove your result.

6. Prove that for all $n \in \mathbb{N}$,
$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$$

7. Experiment with the first few values of $n \in \mathbb{N}$ to conjecture a formula for the value of
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n$$
   Then prove your conjecture using mathematical induction.

8. In a party with at least two people, every person shakes hands with the people they know. Any two given people will either not shake hands or they will shake hands exactly once. Show that there will always be at least one pair of people who shake the same number of hands.

9. **Extra Credit:** For each natural number $m > 1$, find the first $n \in \mathbb{N}$ for which

$$m^n < n!$$

You should prove your answer.