Answers to First Quiz for CSI35

1. Prove that for all natural numbers $n \geq 1$ we have:

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution. We will proceed by induction. For $n = 1$ the statement is:

$$1^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}$$

which is true. This completes the basic step.

Now, assume that the statement has been proven for n, i.e. assume proven that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

We need to prove that

$$\sum_{i=1}^{n+1} i^2 = \frac{(n+1)\left((n+1)+1\right)\left(2(n+1)+1\right)}{6}$$

or equivalently,

$$\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

Indeed, we have:

$$\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^{n} i^2 + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\
&= \frac{(n+1)\left(n(2n+1) + 6(n+1)\right)}{6} \\
&= \frac{(n+1)\left(2n^2 + n + 6n + 6\right)}{6} \\
&= \frac{(n+1)\left(2n^2 + 7n + 6\right)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6}
\end{aligned}$$

$\square$

2. Prove that for all $n \in \mathbb{N}$, 5 divides $n^5 - n$.

Solution. We prove it by induction. For $n = 0$ the statement is

$$5 \mid 0^5 - 0$$

or equivalently

$$5 \mid 0$$

which is true. This completes the basic step. For the inductive step, we assume that it has been proven that $5 \mid n^5 - n$, that is we assume that there is a natural number k such that $n^5 - n = 5k$; and we will prove that $5 \mid (n+1)^5 - (n+1)$. Indeed, we have:

$$\begin{aligned}(n+1)^5 - (n+1) &= n^5 + 5n^4 + 10n^2 + 5n + 1 - n - 1 \\ &= n^5 - n + 5(n^4 + 2n^2 + n) \\ &= 5k + 5(n^4 + 2n^2 + n) \\ &= 5(k + n^4 + 2n^2 + n)\end{aligned}$$

So, $5 \mid (n+1)^5 - (n+1)$. ☐

3. Prove that for all $n \in \mathbb{N}$, 6 divides $n^3 - n$.

   Solution. For this problem we will need the fact that for all natural numbers n, the number $n^2 + n$ is even. This has already been proven in class so it can be assumed known.

   The proof is by induction. For $n = 0$, the statement is true since 6 divides 0. Now we assume that the statement is proven for n, that is we assume that there is a $k \in \mathbb{N}$ such that $n^3 - n = 6k$; and we will prove that $6 \mid (n+1)^3 - (n+1)$. Indeed, we have:

$$\begin{aligned}(n+1)^3 - (n+1) &= n^3 + 3n^2 + 3n + 1 - n - 1 \\ &= n^3 - n + 3(n^2 + n) \\ &= 6k + 3(2l) \\ &= 6k + 6l \\ &= 6(k + l)\end{aligned}$$

   where, to go from the second line to the third, we used the fact that $n^2 + n$ is even and so can be written as $2l$ for some $l \in \mathbb{N}$. Therefore, $6 \mid (n+1)^3 - (n+1)$. ☐

4. Alice and Bob play a game by taking turns removing 1, 2 or 3 stones from a pile that initially has n stones. The person that removes the last stone wins the game. Alice plays always first.

   (a) Prove by induction that if n is a multiple of 4 then Bob has a wining strategy.

   Solution. We proceed by induction. Namely we will prove that for all $k \in \mathbb{N}$, if there are $4k$ stones in the initial pile then Bob has a winning strategy. For $k = 0$, it is Alice's turn to play and there are no stones left, so Bob wins. This completes the basic step. Assume now that the result has been proven for k, that is assume that it has been proven that when there are $4k$ stones in the initial pile, Bob has a winning strategy. To complete the inductive step we need to prove that when there are $4(k+1)$ stones in the initial pile, Bob has a winning strategy. Now, since $4(k+1) = 4k + 4$, after Alice's first move, Bob can always leave $4k$ stones in the pile. Indeed, if Alice takes 1 stone, he takes 3; if she takes 2, he also takes 2; and if she takes 3 he takes 1. With this strategy, Bob can ensure that after his first move, in total 4 stones have been removed from the initial $4k+4$, thus leaving $4k$ stones in the pile. From that point on, Bob can follow the strategy of the second player for a game with $4k$ stones in the initial pile. Therefore Bob has a winning strategy, and this concludes the inductive step. $\square$

(b) Prove that if n is not a multiple of 4 then Alice has a wining strategy.

Solution. If n is not a multiple of 4, then $n = 4k+1$, or $n = 4k+2$, or $n = 4k+3$ for some $k \in \mathbb{N}$. Then Alice in her first move can take 1, or 2, or 3 stones respectively, leaving a pile with $4k$ stones and Bob's turn to play. Now in effect she is the second player in a game that starts with $4k$ stones. Therefore according to part (a) she has a winning strategy. $\square$

5. Chris and Dominique play a slightly different game. Again each player takes turns removing 1, 2 or 3 stones from a pile that initially has n stones but now, the person that removes the last stone loses the game. Chris plays always first. Analyze this game, that is, find the values of n for which Chris has a winning strategy and the values of n for which Dominique has a winning strategy. You should prove your result.

Solution. Dominique has a winning strategy *if and only if* $n = 4k+1$ for some $k \in \mathbb{N}$.

We first prove the *if* part, by induction on k. For $k = 0$ we have a game with 1 stone and Chris's turn. He has to take the stone and therefore Dominique wins. This completes the basic step of the inductive proof. Now we assume that Dominique has a winning strategy when $n = 4k+1$ and we will prove that she has a winning strategy when $n = 4(k+1)+1$ as well. In a game with $4(k+1)+1 = 4k+5$, if Chris takes 1 Dominique takes 3, if he takes 2 she takes 2, and if he takes 3 she takes 1 thus ensuring that after the first two moves 4 stones have been removed in total, leaving a pile with $4k+1$ stones. She can then follow the strategy guaranteed by the inductive hypothesis. Therefore Dominique has a winning strategy, and this concludes the proof of the inductive step.

To prove the *only if* part, we need to prove that if the number of stones in the initial pile does not leave remainder 1 when divided by 4, then Chris has a winning strategy. So we need to prove that if $n = 4k$, or $n = 4k+2$, or $n = 4k+3$, then Chris has a winning strategy. Indeed the first move of Chris is to take 3 or 1 or 2 stones respectively, leaving a pile with a number of stones that leave remainder 1 when divided by 4, and Dominique's turn to play. In effect he is then the second player in a game with $4k+1$ stones, and therefore he has a winning strategy. $\square$

6. Prove that for all $n \in \mathbb{N}$,

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$$

Solution. For $n = 0$ the statement is

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^0 = \begin{pmatrix} a^0 & 0 & 0 \\ 0 & b^0 & 0 \\ 0 & 0 & c^0 \end{pmatrix}$$

or equivalently,

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is true. This completes the inductive step.

Now we assume that the statement has been proven for n and we will prove it for $n+1$, that is we will prove that

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^{n+1} = \begin{pmatrix} a^{n+1} & 0 & 0 \\ 0 & b^{n+1} & 0 \\ 0 & 0 & c^{n+1} \end{pmatrix}$$

We have

$$\begin{aligned} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^{n+1} &= \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^{n} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \\ &= \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \\ &= \begin{pmatrix} a^{n+1} & 0 & 0 \\ 0 & b^{n+1} & 0 \\ 0 & 0 & c^{n+1} \end{pmatrix} \end{aligned}$$

This concludes the inductive step. □

7. Experiment with the first few values of $n \in \mathbb{N}$ to conjecture a formula for the value of

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n$$

Then prove your conjecture using mathematical induction.

Solution. We will prove that

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

We proceed by mathematical induction. For $n = 0$ the statement is:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^0 = \begin{pmatrix} 1 & 0 & \frac{0(0+1)}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or equivalently,

$$\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}^0 = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

which is true. This completes the basic step.

Now we assume that the statement is true for n and prove it for $n+1$. We have:

$$\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}^{n+1} = \begin{pmatrix}1 & n & \\ 0 & 1 & n\\ 0 & 0 & 1\end{pmatrix}^n \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & n & \frac{n(n+1)}{2}\\ 0 & 1 & n\\ 0 & 0 & 1\end{pmatrix}^n \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & n+1 & 1+n+\frac{n(n+1)}{2}\\ 0 & 1 & n+1\\ 0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & n+1 & \frac{n(n+1)+2(n+1)}{2}\\ 0 & 1 & n+1\\ 0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & n+1 & \frac{(n+1)(n+2)}{2}\\ 0 & 1 & n+1\\ 0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & n+1 & \frac{(n+1)((n+1)+1)}{2}\\ 0 & 1 & n+1\\ 0 & 0 & 1\end{pmatrix}$$

So we have shown that

$$\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}^{n+1} = \begin{pmatrix}1 & n+1 & \frac{(n+1)((n+1)+1)}{2}\\ 0 & 1 & n+1\\ 0 & 0 & 1\end{pmatrix}$$

which completes the inductive step. □

8. In a party with at least two people, every person shakes hands with the people they know. Any two given people will either not shake hands or they will shake hands exactly once. Show that there will always be at least one pair of people who shake the same number of hands.

   Solution. Let n be the number of people in the party. We will prove the statement using induction on n. When $n = 2$ there are only two people in the party and they either shake hands or they don't. In the first case each has shaken one hand while in the latter case each has shaken 0 hands; in any case they have shaken the same number of hands. This concludes the proof of the basic step.

Now we assume that the statement is true for n and we will prove that it is also true for $n+1$. So consider a party with $n+1$ people. If there is a person that doesn't shake any hands, the remaining n people shake hands "among themselves". Therefore, by the inductive hypothesis, there is at least a pair of those n that have shaken the same number of hands. On the other hand, if there is no person that has shaken 0 hands, each of the $n+1$ people has shaken, 1, or 2, $\ldots$, or n hands. So we have $n+1$ people and n possibilities for how many hands each has shaken. Since there aren't enough possibilities for each person to have shaken a different number of hands it follows that at least two people have shaken the same number of hands. It follows that in any case, there are at least two people that have shaken the same number of hands, and this concludes the inductive step. $\square$