Final exam for CSI35

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1. Let n be a positive integer. Prove that:

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right)=n+1$$

2. For an integer n let $g(n)$ be the number of ways that n can be written as a sum of ones, twos, and threes, where the order that the summands are written is important. For example, $g(1)=1$, $g(2)=2$ since 2 can be written either as 2 or as $1+1$, and $g(3)=4$ because 3 can be written as $1+1+1$ or as $1+2$ or as $2+1$ or as 3.

   (a) Find a recursive definition of $g(n)$

   (b) Prove that this recursive definition is correct.

3. This question is about the Tower of Hanoi puzzle. The puzzle consists of three pegs and six disks, initially stacked in decreasing size on one of the pegs, say A, as shown in Figure 1.

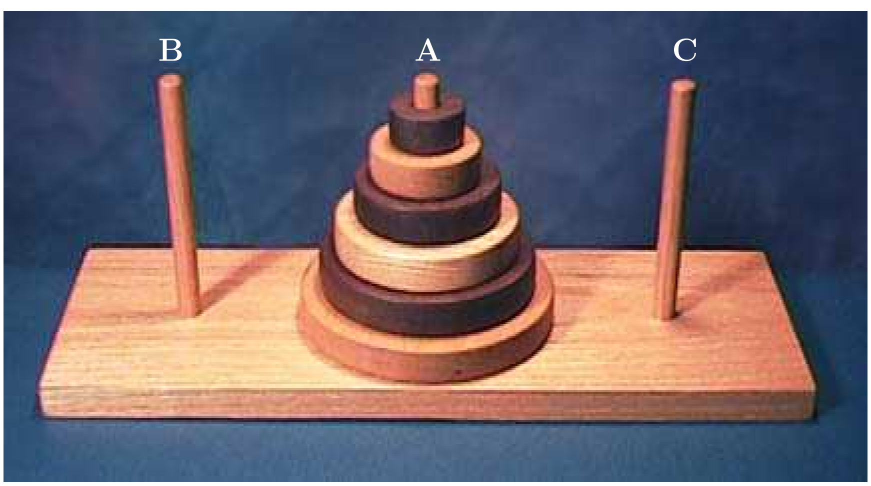

Figure 1: The Towers of Hanoi puzzle

   The goal is to transfer the whole tower of six disks to one of the other pegs, say C, moving only one disk at a time and never moving a larger one on top of a smaller.

   (a) Prove that this puzzle has a solution for any number of initial disks.

   (b) Prove that the puzzle with n disks can be solved in 2^n-1 moves.

   (c) Prove that the puzzle with n disks cannot be solved in fewer than 2^n-1 moves.

4. Let $\mathbb{N}^*$ be the set of positive integers. The relation $\sim$ on $\mathbb{N}^*$ is defined as follows:

$$m \sim n \Longleftrightarrow \exists k \in \mathbb{N}^* \quad mn = k^2$$

   (a) Prove that $\sim$ is an equivalence relation.

   (b) Find the equivalence classes of 2, 4, and 6.

5. If the relation $\sim$ of Question 4 was defined on $\mathbb{N}$ instead of $\mathbb{N}^*$ would it still be an equivalence relation? Prove your answer.

6. Consider the poset $(\mathcal{P}(\{1,2,3,4,5\}), \subseteq)$.

   (a) Draw the Hasse diagram of this poset.

   (b) Find the greatest lower bound and the least upper bound of the set $\{\{2,3\},\{3,4,5\}\}$ if they exist.

7. Can you find a graph with six vertices and degrees $1,2,3,4,5,6$?

8. For a natural number n let Q_n denote the n-cube graph.

   (a) How many vertices and how many edges does Q_n have? Prove your answers.

   (b) Prove that Q_n is bipartite.

9. Are the two graphs in Figure 2 isomorphic? How about the graphs in Figure 3? Prove your answers.

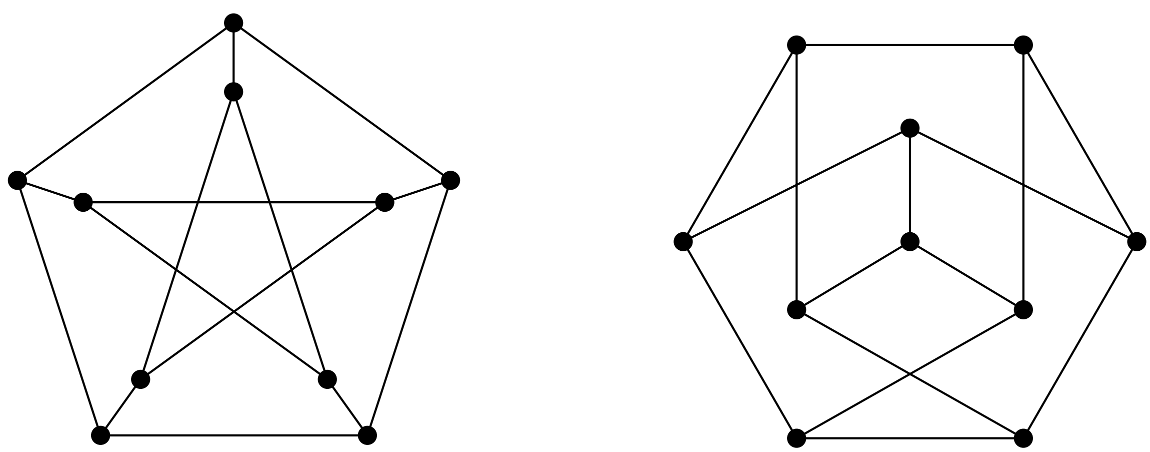

Figure 2: The first two graphs of Question 9

10. Count von Diamond has been murdered in his estate. The internationally known detective (and part time graph theorist) Inspector Clouseau has been called in to investigate. The butler claims that he saw the gardener enter the pool room (where the murder took place) and then, shortly after, leave that room by the same door. On the other hand, the gardener says that he cannot be the man that the butler saw because he entered the house, went through each door exactly once and then left the house. Inspector Clouseau checks the floor plan (see Figure 4) and within minutes declares the case solved. Who done it?

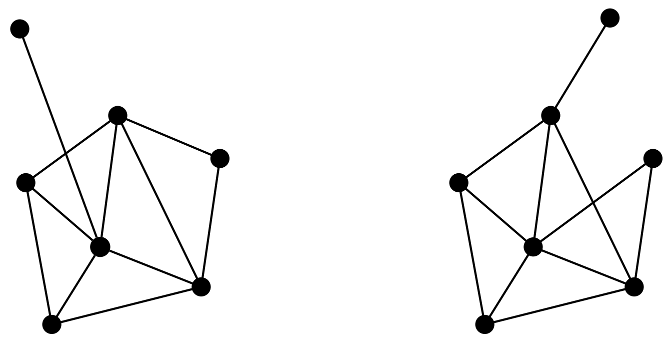

Figure 3: The last two graphs of Question 9

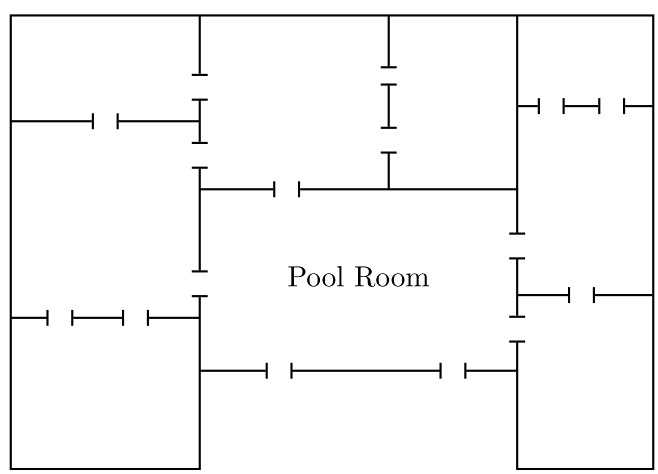

Figure 4: The floor of Von Diamond Estate

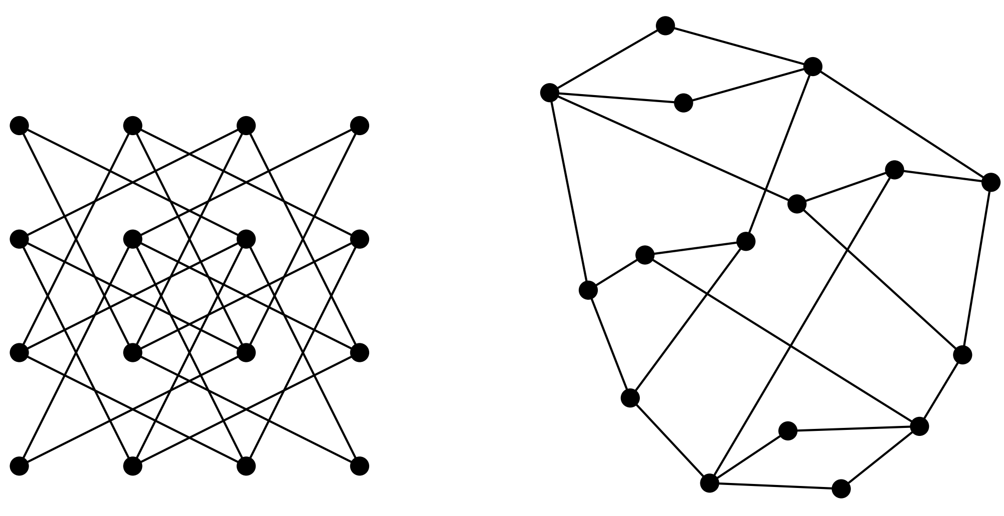

Figure 5: The graphs of Question 12

11. For which values of n does the complete graph K_n have an Euler circuit?
12. Prove that the two graphs in Figure 5 are isomorphic.
13. Prove that the graph in the right side of Figure 5 is not Hamiltonian.
14. Prove that the 4×4 chessboard does not admit a knight's tour.
15. List all possible trees with six or less vertices.
16. **Extra Credit** Find a knight's tour on an 8×8 chessboard.

    Hint. There are many knight's tours on the standard chessboard. One way to proceed (invented by our old acquaintance Leonard Euler) is to first find an open knight's tour on a 4×8 board that starts and ends at the top row; to find a knight's tour on the 8×8 board then you just "glue" that open tour and its mirror image.