Midterm exam for CSI35

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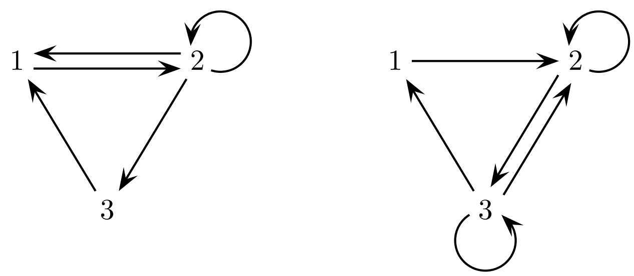
1. The Fina Bocci company breeds worms for fishing. After each worm is two weeks old they cut off its tail which becomes a new worm. The tail grows back in a week, so once a worm becomes two weeks old it produces a new worm every week.
   (a) Assuming that no worm ever dies and that the company starts with one newly "born" worm, find a recursive formula for the the number of worms after n weeks.
   (b) Prove that the formula you found in part a) is correct.
2. Let f_n denote the n-th Fibonacci number where n is a natural number. Show that for all $n \in \mathbb{N}$
$$f_{n-1}f_{n+1} - f_n^2 = (-1)^n$$
3. Recall that a palindrome is a string that equals its reverse, in other words a string that reads the same when read backwards. Find a formula for the number of palindromes of length n that can be constructed using only letters from the alphabet $\{a, b, c\}$.
4. Prove that for all natural numbers $n \geq 1$:
$$1^3 + 2^3 + \cdots n^3 = \left(\frac{n(n+1)}{2}\right)^2$$
5. Alice and Bob play the following game: they take turns removing 1 to 5 stones from a pile of initial size n. The first person who cannot play because there are no stones to take, loses the game. Alice always plays first.
   (a) Prove that if n is a multiple of 6 then Bob has a winning strategy.
   (b) Prove that if n is not a multiple of 6 then Alice has a winning strategy.
6. In Nevereverland they have only stamps worth 5 or 7 cents. Prove that a nevereverlander can send any letter that costs 24 cents or more.
7. Consider the following zero-one matrix:
$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$
   Prove that $A^n = A$ for all natural numbers $n \geq 1$, where the power is with respect to the boolean product.

8. Consider the digraphs G_1 and G_2 shown bellow:



G_1 $\qquad$ G_2

Draw the digraph $G_1 \circ G_2$.

Hint. Consider the corresponding boolean matrices.

9. Prove that
$$R = \{(a, b) | a^4 = b^4\}$$
is an equivalence relation on $\mathbb{R}$ and find the equivalence class of -4.

10. Consider the following relation on the set $\mathbb{Z} \times \mathbb{Z}^*$, where $\mathbb{Z}^*$ denotes the set of non-zero integers:
$$((m, n), (k, l)) \in R \iff ml = kn$$
   (a) Prove that R is an equivalence relation.
   (b) Find the equivalence class of $(2, 4)$.

11. After Alice and Bob finished playing the game in Problem 5 they decided to study for the Discrete Mathematics exam that was coming up. After a while they had the following conversation:

A: I was looking at the definition of an equivalence relation and it seems redundant.

B: How so?

A: Well, it says that an equivalence relation is a relation that is reflexive, symmetric and transitive. Now that's redundant, because I can prove that if a relation is symmetric and transitive then it is reflexive as well. So to be economical we should define an equivalence relation to be a relation that is symmetric and transitive. No need to check for reflexivity, really.

B: Hm, this sound fishy to me. Somebody would've noticed before. Let me see your proof.

A: It's very simple really: Let R be a symmetric and transitive relation on a set A. To prove that it is reflexive I need to prove that for all $x \in A$ we have $(x, x) \in R$. So let $x \in R$, chose any $y \in A$ such that $(x, y) \in R$. Then since R is symmetric we have $(y, x) \in R$ also. So we have $(x, y) \in R$ and $(y, x) \in R$, so since R is transitive we conclude that $(x, x) \in R$. So, R is reflexive.

Bob is thoughtful for a while.

B: Hm, your proof seems valid... Still, I find it hard to believe that nobody had noticed this before. Let me think some more.

Bob thinks some more.

B: Ok, your proof must be wrong because I can produce a counter example. Remember the matrix in exercise 7:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

This matrix is symmetric and transitive, but not reflexive.

A: (after some thought) Yep, you're right. Still though I can't figure out where is the gap in my proof.

B: Me neither . . .

(a) Prove that Bob's counter-example really works.

(b) Can you find the fault in Alice's proof?