Fifth Quiz for Math 30, section 6432
The answers

Note: In what follows I use $p(x)$ to stand for the polynomial in question.

1. Use Descartes's rule of signs to determine the possible number of positive and negative zeros of the following polynomials:

   (a) $x^3 + 2x^2 + 3x + 4$

   Answer. Since there is no change in signs (that is 0 sign changes) this polynomial at most 0 positive roots. That is, this polynomial has *no* positive roots.

   To find the number of possible negative roots we calculate $p(-x)$:

   $$\underbrace{-x^3+}_{1}\underbrace{2x^2-}_{2}\underbrace{3x+}_{3}4$$

   There are three changes of sign, so this polynomial has 3 or 1 negative roots. □

   (b) $3x^4 - 3x^3 + 2x^2 + 4x + 7$

   Answer. To find the number positive roots we count the number of sign changes in the coefficients of $p(x)$:

   $$\underbrace{+3x^4-}_{1}\underbrace{3x^3+}_{2}2x^2 + 4x + 7$$

   So there are 2 or 0 positive roots.

   To find the number of negative roots we count the sign changes in $p(-x)$:

   $$3x^4 + 3x^3\underbrace{+2x^2-}_{1}\underbrace{4x+}_{2}7$$

   So there are 2 or 0 negative roots. □

   (c) $-5x^5 - 4x^4 + 3x^3 + 2x^2 + x + 23$

   Answer. We have:

   $$-5x^5\underbrace{-4x^4+}_{1}3x^3 + 2x^2 + x + 23$$

   So this polynomial has 1 positive root.

   Looking at $p(-x)$ we have:

   $$\underbrace{+5x^5-}_{1}4x^4\underbrace{-3x^3+}_{2}\underbrace{2x^2-}_{3}\underbrace{x+}_{4}23$$

   So $p(x)$ has 4 or 2 or 0 negative roots. □

   (d) $-5x^5 + x^4 - 3x^3 - 10x^2 + 29x - 32$

Answer. We have:

$$-\underbrace{5x^5+}_{1}\underbrace{x^4-}_{2}3x^3\underbrace{-10x^2+}_{3}\underbrace{29x-}_{4}32$$

So there are 4 or 3 or 2 or 0 positive roots.

Examining $p(-x)$ we have:

$$5x^5 + x^4\underbrace{+3x^3-}_{1}10x^2 - 29x - 32$$

So $p(x)$ has one negative root. □

2. Prove that the following polynomial has at least two non-real roots:

$$2x^7 - 11x^6 - 71x^5 + 450x^4 + 1740x^3 + 1189x^2 + 728$$

Answer. Using Descartes's rule of signs we have:

$$\underbrace{+2x^7-}_{1}11x^6\underbrace{-71x^5+}_{2}450x^4 + 1740x^3+1189x^2 + 728$$

So $p(x)$ has at most 2 positive roots.

Also looking at $p(-x)$ we have:

$$-2x^7\underbrace{-11x^6+}_{1}71x^5\underbrace{+450x^4-}_{2}\underbrace{1740x^3+}_{3}1189x^2 + 728$$

So $p(x)$ has at most 3 negative roots. Therefore $p(x)$ has at most 5 real roots (since 0 is not a root of $p(x)$).

Now, according to the Fundamental Theorem of Algebra, $p(x)$ has exactly 7 complex roots. Since at most 5 of those roots are real we conclude that there is at least two complex non-real roots. □

3. For each of the following rational functions find the domain, possible x and y intercepts as well as all possible asymptotes.

   (a) $f(x) = \dfrac{2x+2}{x^2-3x-4}$

   Answer. We first factor numerator and denominator:

$$f(x) = \frac{2(x+1)}{(x+1)(x-4)}$$

The roots of the denominator have to be excluded from the domain of f. Therefore the domain is:

$$(-\infty, -1) \cup (-1, 4) \cup (4, \infty)$$

Since $x+1$ is a common factor of the numerator and denominator, for x in the domain of f we have:

$$f(x) = \frac{2}{x-4}$$

It follows that $y = -2$ a vertical asymptote to the graph of $y = f(x)$. Also since the denominator has larger degree than the numerator the x–axis, $y = 0$ is a horizontal asymptotes.

To find the y–intercept we put $x = 0$ at the formula to get $f(0) = -\frac{1}{2}$, so the y–intercept is $\left(0, \frac{1}{2}\right)$.

Since the numerator never vanishes the graph has no x–intercept. □

(b) $f(x) = \dfrac{x^2+x-6}{x^3+3x^2-4x}$

Answer. We first factor the numerator and denominator:

$$f(x) = \frac{(x+3)(x-2)}{x(x+4)(x-1)}$$

The zeros of the denominator have to be excluded from the domain so the domain is

$$(-\infty,-4) \cup (-4,0) \cup (0,1) \cup (1,\infty)$$

Since the numerator and denominator have no common factors we have vertical asymptotes at all roots of the denominator. So we have three vertical asymptotes: $x = -4$, $x = 0$, and $x = 1$.

The degree of the denominator is larger than the degree of the numerator and therefore the x–axis, $y = 0$ is a horizontal asymptote.

Since 0 is not in the domain of f there is no y–intercept.

The x–intercepts are given by the roots of the numerator. So we have two x–intercepts: $(-3,0)$ and $(2,0)$. □

(c) $g(x) = \dfrac{x^2+2x+5}{x+2}$

Answer. The root of the denominator has to be excluded from the domain. Therefore the domain is:

$$(\infty,-2) \cup (-2,\infty)$$

We have a vertical asymptote: $x = -2$.

Since the degree of the numerator is 1 more than the degree of the denominator we have a slant asymptote. To find its equation we perform the division (we use synthetic division):

-2	1	2	5
		-2	0
	1	0	5

Thus the equation of the slant asymptote is $y = x$.

To find the y–intercept we substitute $x = 0$ in the formula and we get $x = \frac{5}{2}$. Thus the y–intercept is $\left(0, \frac{5}{2}\right)$.

Since the numerator never vanishes there are no x–intercepts. □

(d) $h(x) = \dfrac{3x^2 - 9x + 6}{2x^2 + 6x + 4}$

Answer. We first factor numerator and denominator:

$$h(x) = \frac{3(x-1)(x-2)}{2(x+1)(x+2)}$$

The roots of the denominator have to be excluded from the domain, so we have that the domain of h is:

$$(-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$$

We have vertical asymptotes at the roots of the denominator. So we have two asymptotes $x = -2$ and $x = -1$.

The numerator and denominator have the same degree so we have a horizontal asymptote $y = \frac{3}{2}$.

To find the y–intercept we put $x = 0$ and we get $h(0) = \frac{3}{2}$ so the y–intercept is $(0, \frac{3}{2})$.

We have x–intercepts at the roots of the numerator. So we have two x–intercepts: $(1, 0)$ and $(0, 2)$. □

4. Solve the following inequality using the "graphing method".

$$x^4 + 4x^3 + 3x^2 \geq 4x + 4$$

Answer. We first write the equations so that the right hand side is 0.

$$x^4 + 4x^3 + 3x^2 - 4x - 4 \geq 0$$

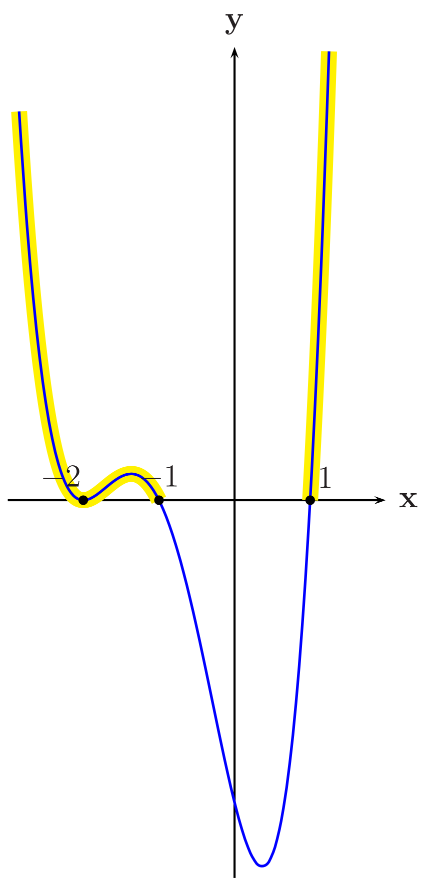
Then we factor the left hand side. The possible rational roots are $\{\pm 1, \pm 2, \pm 4\}$. So we try:

1	1	4	3	-4	-4
		1	5	8	4
-1	1	5	8	4	0
		-1	-4	-4	
	1	4	4	0	

So we have:

$$x^4 + 4x^3 + 3x^2 - 4x - 4 = (x-1)(x+1)(x^2+4x+4)$$
$$= (x-1)(x+1)(x+2)^2$$

It follows that the graph of $y = x^4 + 4x^3 + 3x^2 - 4x - 4$ looks like the following figure:



So the solution is

$$(-\infty, -1] \cup [1, \infty)$$

□

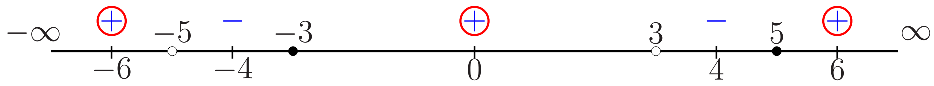
5. Solve the following inequality using the "test points" method.

$$\frac{x^2 - 2x - 15}{x^2 + 2x - 15} \geq 0$$

Answer. After factoring numerator and denominator we get

$$\frac{(x+3)(x-5)}{(x+5)(x-3)} \geq 0$$

So we have the real line breaking in to subintervals as follows:



Let

$$f(x) := \frac{(x+3)(x-5)}{(x+5)(x-3)}$$

Then choosing suitable test points we get:

$$\begin{aligned} f(-6) &= \frac{11}{3} > 0 \\ f(-4) &= \frac{-9}{7} < 0 \\ f(0) &= 1 > 0 \\ f(4) &= \frac{-7}{9} < 0 \\ f(6) &= \frac{3}{11} > 0 \end{aligned}$$

So the solution is

$$(-\infty, -5) \cup [-3, 3) \cup [5, \infty)$$

□

6. Solve the following inequality using the "table of signs" method.

$$\frac{(x-1)(x+1)(x+2)}{x-2} < 0$$

Solution. We have the following table of signs:

	$-\infty$	-2		-1		1		2	∞
$x+2$	$-$	0	$+$		$+$		$+$		$+$
$x+1$	$-$		$-$	0	$+$		$+$		$+$
$x-1$	$-$		$-$		$-$	0	$+$		$+$
$x-2$	$-$		$-$		$-$		$-$	0	$+$
$f(x)$	$+$	0	$-$	0	$+$	0	$-$		$+$

So the solution is:

$$(-2, -1) \cup (1, 2)$$

□