Answers to the second exam

1. Use Descartes's rule of signs to determine the possible number of positive and negative roots for the function

$$f(x) = 5x^4 - x^5 + 5x^2 - 7x - 1$$

Answer. We first put the polynomial in decreasing order:

$$f(x) = \underbrace{-x^5 +}_{1} 5x^4 \underbrace{+5x^2 -}_{2} 7x - 1$$

Therefore $f(x)$ has 2 or 0 positive roots. On the other hand:

$$f(-x) = x^5 + 5x^4 + 5x^2 \underbrace{+7x -}_{1} 1$$

Therefore $f(x)$ has 1 negative root. □

2. Find a rational function that has has two vertical asymptotes $x = 2$ and $x = 3$, x–intercepts at $x = -3$ and $x = 1$ and a horizontal asymptote $y = 3$.

Answer. Vertical asymptotes occur at the root of the denominator so the denominator has to have $(x - 2)$ and $(x - 3)$ as factors. So we can take the denominator to be

$$(x - 2)(x - 3) = x^2 - 5x + 6$$

The x–intercepts of a rational function are given by the roots of its numerator. So the numerator has to have $(x+3)$ and $(x-1)$ as factors. Furthermore in order to have a horizontal asymptote at $x = 3$ the numerator has to have the same degree as the denominator and the quotient of the leading terms of the numerator and denominator should be 3. So we can take the numerator to be:

$$3(x + 3)(x - 1) = 3x^2 + 6x - 9$$

In sum we have the function:

$$f(x) = \frac{3x^2 + 6x - 9}{x^2 - 5x + 6}$$

□

3. For each of the following functions find the domain, possible x and y intercepts, and all asymptotes.

   (a) $f(x) = \dfrac{x^2 - 6x + 5}{x^3 - x^2 - 2x}$

   Answer. We first factor numerator and denominator:

$$f(x) = \frac{(x - 1)(x - 5)}{x(x + 1)(x - 2)}$$

   The domain consists of those x that do not make the denominator 0. So the roots of the denominator have to be excluded from the domain. Now the denominator vanishes at $x = 0$, $x = -1$, and $x = 2$. Therefore the domain is

$$(-\infty, -1) \cup (-1, 0) \cup (0, 2) \cup (2, \infty)$$

The y–intercept is obtained by evaluating the function at $x = 0$. Since however 0 is not on the domain of f there is no y–intercept.

The x–intercepts occur at the roots of the numerator, so we have x–intercepts at the points $(1, 0)$ and $(5, 0)$.

We have vertical asymptotes at the roots of the denominator. So we have three vertical asymptotes $x = -1$, $x = 0$, and $x = 2$.

Since the denominator has larger degree than the numerator the y–axis is also a horizontal asymptote. □

(b) $g(x) = \dfrac{x^2 + 2x - 3}{x - 3}$

Answer. We first factor:

$$g(x) = \frac{(x+3)(x-1)}{x-3}$$

The domain consists of those x that do not make the denominator 0. So the roots of the denominator have to be excluded from the domain. Now the denominator vanishes only at $x = 3$. Therefore the domain is

$$(-\infty, -3) \cup (3, \infty)$$

The y–intercept is obtained by evaluating the function at $x = 0$. Since $g(0) = 1$ we have a y–intercept at $(0, 1)$.

The x–intercepts occur at the roots of the numerator. So we have x–intercepts at the points $(-3, 0)$ and $(1, 0)$.

We have vertical asymptotes at the roots of the denominator. So we have one vertical asymptote $x = 3$.

Since the degree of the numerator is one more than the degree of the denominator we also have a slant asymptote. To get the equation of the slant asymptote we perform the division:

$$\begin{array}{r|rrr} 3 & 1 & 2 & -3 \\ & & & \\ & & 3 & 15 \\ \hline & 1 & 5 & 12 \end{array}$$

So the slant asymptote has equation $y = x + 5$. □

4. Solve the inequality:

$$(x+2)^2(x+3)(x-1)^3 < 0$$

Answer. We'll use the "table of signs" method. We have the following table:

	$-\infty$	-3		-2		1	∞
$x+3$	$-$	0	$+$		$+$		$+$
$(x+2)^2$	$+$		$+$	0	$+$		$+$
$(x-1)^3$	$-$		$-$		$-$	0	$+$
$f(x)$	$+$	0	$-$	0	$-$	0	$+$

So the solution set is

$$(-3,-2)\cup(-2,1)$$

□

5. Solve the inequality:

$$\frac{x^2+3x-10}{x+3}\geq 0$$

Answer. We first factor:

$$\frac{(x+5)(x-2)}{x+3}\geq 0$$

So we have the following table:

	$-\infty$	-5		-3		2	∞
$x+5$	$-$	0	$+$		$+$		$+$
$x+3$	$-$		$-$	0	$+$		$+$
$x-2$	$-$		$-$		$-$	0	$+$
$f(x)$	$-$	0	$+$		$-$	0	$+$

So the solution set is

$$[-5,-3)\cup[2,\infty)$$

□

6. Simplify as much as possible:

$$\log_2 4x^3-2\log_2 x+\log_2\left(\frac{1}{x}\right)$$

Answer.

$$\begin{aligned}\log_2 4x^3 - 2\log_2 x + \log_2\left(\frac{1}{x}\right) &= \log_2 4 + \log_2 x^3 - 2\log_2 x - \log_2 x\\ &= 2 + 3\log_2 x - 2\log_2 x - \log_2 x\\ &= 2\end{aligned}$$

□

7. Find the domain of the following function:

$$f(x) = \ln(4 - 12x)$$

Answer. The argument of the logarithm should be positive. Therefore the domain consists of those x that satisfied the inequality:

$$4 - 12x > 0 \Longleftrightarrow 4 > 12x \Longleftrightarrow \frac{1}{3} > x$$

So the domain is $(-\infty, \frac{1}{3})$ □

8. Solve for x:

$$\log_3 x + \log_3(x+2) = 1$$

Answer.

$$\begin{aligned}\log_3 x + \log_3(x+2) = 1 &\Longrightarrow \log_3 x(x+2) = 1\\ &\Longleftrightarrow 3^{\log_3 x(x+2)} = 3^1\\ &\Longleftrightarrow x(x+2) = 3\\ &\Longleftrightarrow x^2 + 2x = 3\\ &\Longleftrightarrow x^2 + 2x - 3 = 0\\ &\Longleftrightarrow (x+3)(x-1) = 0\\ &\Longleftrightarrow x = -3 \text{ or } x = 1\end{aligned}$$

The solution $x = 1$ is acceptable since it satisfies the original equation. However $x = -3$ is not acceptable since we cannot substitute $x = -3$ in the original equation. Therefore the only solution is

$$x = 1$$

□

9. Solve for x:

$$4^{3x-4} = 16$$

Answer.

$$\begin{aligned}4^{3x-4} = 16 &\Longleftrightarrow \log_4 4^{3x-4} = \log_4 16\\ &\Longleftrightarrow 3x - 4 = 2\\ &\Longleftrightarrow 3x = 6\\ &\Longleftrightarrow x = 2\end{aligned}$$

□

10. Find the inverse of the following function:

$$f(x) = 2^{3x+5}$$

Answer. To find the inverse we start with the equation $y = f(x)$, we interchange x and y and then solve for y. So we have:

$$\begin{aligned} x = 2^{3y+5} &\iff \log_2 x = \log_2 2^{3y+5} \\ &\iff \log_2 x = 3y + 5 \\ &\iff \log_2 x - 5 = 3y \\ &\iff \frac{\log_2 x - 5}{3} = y \end{aligned}$$

Therefore

$$f^{-1}(x) = \frac{\log_2 x - 5}{3}$$

□