Answers to the first exam

1. Graph $y = x^2 + 2x - 3$. Possible x or y intercepts should be identified exactly.

   Answer. We have:

$$\begin{aligned} x^2 + 2x - 3 &= x^2 + 2x + 1 - 1 - 3 \\ &= (x^2 + 2x + 1) - 4 \\ &= (x+1)^2 - 4 \end{aligned}$$

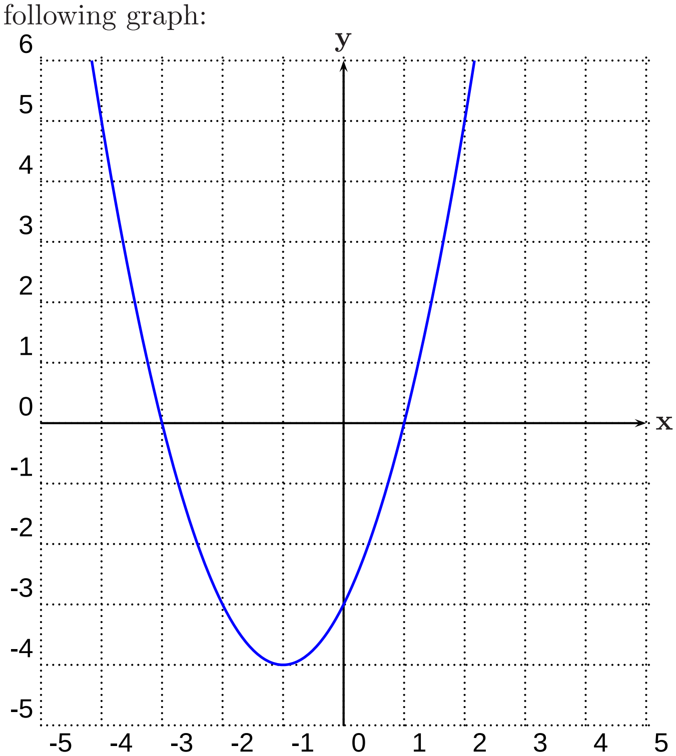
   So the vertex of the parabola is at the point $(-1, -4)$. The parabola is concave upwards since the leading coefficient is positive. To find the x–intercepts we solve:

$$x^2 + 2x - 3 = 0 \iff x = -3 \quad \text{or} \quad x = 1$$

   The y–intercept is obtained by setting $x = 0$ in the equation. So we have:

$$y(0) = -3$$

   In sum we have the following graph:



□

2. Find the domain, the range, and the inverse function of the following function:

$$f(x) = \frac{3}{x-5}$$

   Your answers should be in interval notation.

Answer. The formula of the function involves a denominator so we have to exclude the roots of the denominator from the domain. So we must have:

$$x - 5 \neq 0 \iff x \neq 5$$

Using interval notation we have that the domain is

$$(-\infty, 5) \cup (5, \infty)$$

The range of f equals the domain of its inverse. So we first calculate the inverse function. We start with the equation for f:

$$y = \frac{3}{x-5}$$

we interchange x and y:

$$x = \frac{3}{y-5}$$

and then solve for y:

$$\begin{aligned} x = \frac{3}{y-5} &\iff x(y-5) = 3 \\ &\iff xy - 5x = 3 \\ &\iff xy = 5x + 3 \\ &\iff y = \frac{5x+3}{x} \end{aligned}$$

Therefore the inverse function has formula:

$$f^{-1}(x) = \frac{5x+3}{x}$$

The denominator has to be different than 0 so the domain of f^{-1} is:

$$(-\infty, 0) \cup (0, \infty)$$

It follows that the range of f is $(-\infty, 0) \cup (0, \infty)$. □

3. Verify that the following two functions are inverses of each other: $f(x) = \dfrac{3x-1}{x+2}$ and $g(x) = \dfrac{2x+1}{3-x}$.

Answer. We will show that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. We start with $f \circ g$:

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= \frac{3\dfrac{2x+1}{3-x} - 1}{\dfrac{2x+1}{3-x} + 2} \\
&= \frac{\dfrac{6x+3}{3-x} - 1}{\dfrac{2x+1}{3-x} + 2} \\
&= \frac{(6x+3) - (3-x)}{(2x+1) + 2(3-x)} \\
&= \frac{6x+3-3+x}{2x+1+6-2x} \\
&= \frac{7x}{7} \\
&= x
\end{aligned}$$

Now $g \circ f$:

$$\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= \frac{2\dfrac{3x-1}{x+2} + 1}{3 - \dfrac{3x-1}{x+2}} \\
&= \frac{\dfrac{6x-2}{x+2} + 1}{3 - \dfrac{3x-1}{x+2}} \\
&= \frac{(6x-2) + (x+2)}{3(x+2) - (3x-1)} \\
&= \frac{6x-2+x+2)}{3x+6-3x+1} \\
&= \frac{7x}{7} \\
&= x
\end{aligned}$$

Therefore f and g form a pair of inverse functions. □

4. List all possible rational solutions of the following equation:

$$2x^5 - 3x^4 + 5x^3 + x^2 - 3x + 12 = 0$$

You don't need to check whether these are actual solutions. Just to list them.

Answer. A rational solution $\dfrac{p}{q}$ will have denominator p that divides the constant term 12 and numerator q that divides the leading coefficient 2. All divisors of 12 are $\pm\{1, 2, 3, 4, 6, 12\}$ and the divisors of 2 are $\pm\{1, 2\}$. So all possible roots, listed with increasing denominator, are:

$$\pm\{1, 2, 3, 4, 6, 12, \frac{1}{2}, \frac{3}{2}\}$$

□

5. (a) Solve the following equation:

$$x^5 + 2x^4 - 6x^3 - 8x^2 + 5x + 6 = 0$$

Answer. All possible rational roots are $\pm\{1, 2, 3, 6\}$. We check them starting with $x = 1$. We use the synthetic division algorithm:

$$\begin{array}{c|cccccc}
1 & 1 & 2 & -6 & -8 & 5 & 6 \\
 & & 1 & 3 & -3 & -11 & -6 \\
\hline
1 & 1 & 3 & -3 & -11 & -6 & 0 \\
 & & 1 & 4 & 1 & -10 & \\
\hline
 & 1 & 4 & 1 & -10 & -16 &
\end{array}$$

So $x = 1$ is a single solution. Next we check $x = -1$:

$$\begin{array}{c|ccccc}
-1 & 1 & 3 & -3 & -11 & -6 \\
 & & -1 & -2 & 5 & 6 \\
\hline
-1 & 1 & 2 & -5 & -6 & 0 \\
 & & -1 & -1 & 6 & \\
\hline
 & 1 & 1 & -6 & 0 &
\end{array}$$

So $x = -1$ is a double solution. The quotient is a quadratic polynomial:

$$x^2 + x - 6$$

that is easily seen (by quadratic formula or factoring) to have the following two solutions:

$$x = -3 \quad \text{or} \quad x = 2$$

In sum we have the following solutions:

$$x = -3 \quad \text{or} \quad x = -1 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = 2$$

and $x = -1$ has multiplicity 2. □

(b) Sketch a rough graph of the function

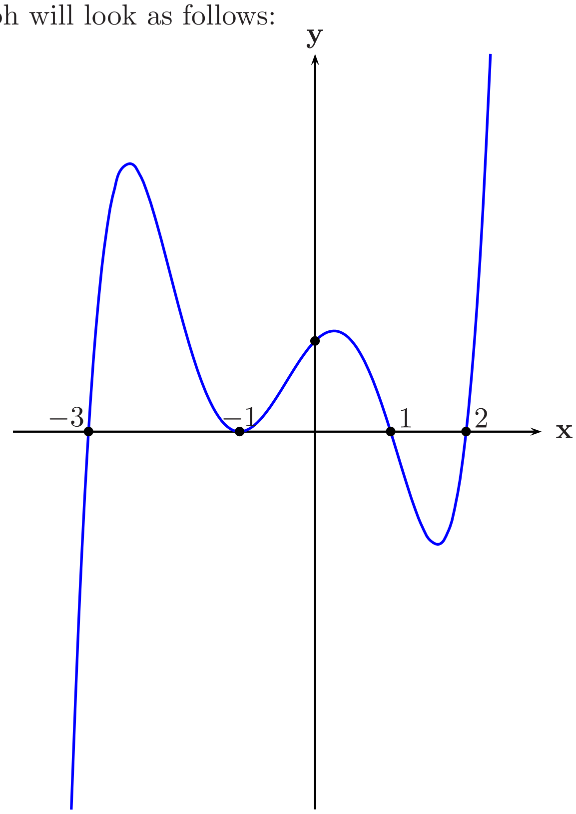
$$f(x) = x^5 + 2x^4 - 6x^3 - 8x^2 + 5x + 6$$

The graph should correctly reflect the end behavior, the behavior near zeros and the number of turning points.

Answer. From the previous part we know that the formula for f can be factored as follows:

$$f(x) = (x+3)(x+1)^2(x-1)(x-2)$$

So the end behavior of f is the same as that of $y = x^5$, the y–intercept is $y = 6$ and the graph cuts the x–axis at $x = -3$, $x = 1$, and $x = 2$ while it touches the x–axis at $x = -1$. So a sketch of the graph will look as follows:



□

6. Let $f(x) = \sqrt{2x+4}$ and $g(x) = x - 1$. Find the domain and the formula for the function $\dfrac{f}{g}$.

Answer. The formula is:

$$\frac{f}{g}(x) = \frac{\sqrt{2x+4}}{x-1}$$

The formula involves a square root and a denominator. So the domain will consist of those real numbers that make the quantity under the radical sign positive *and at the same time* the denominator non-zero. For the radical sign we have:

$$2x + 4 \geq 0 \Longleftrightarrow x \geq -2$$

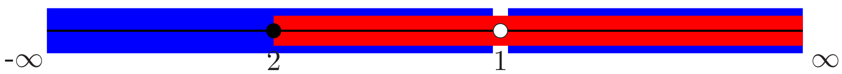
while for the denominator we have

$$x - 1 \neq 0 \Longleftrightarrow x \neq 1$$

Thus the domain will be the following intersection:

$$[2, \infty) \cap ((-\infty, 1) \cup (1, \infty))$$

We perform this calculation graphically:



We see that the domain is:

$$[-2, 1) \cup (1, \infty)$$

□

7. The graph of the function g is obtained by shifting the graph of the function $f(x) = 3x^5$ four units to the right along the x–axis and three units downwards along the y–axis. Find a formula for $g(x)$. (**You don't need to graph** g).

Answer.

$$g(x) = 3(x - 4)^5 - 3$$

□

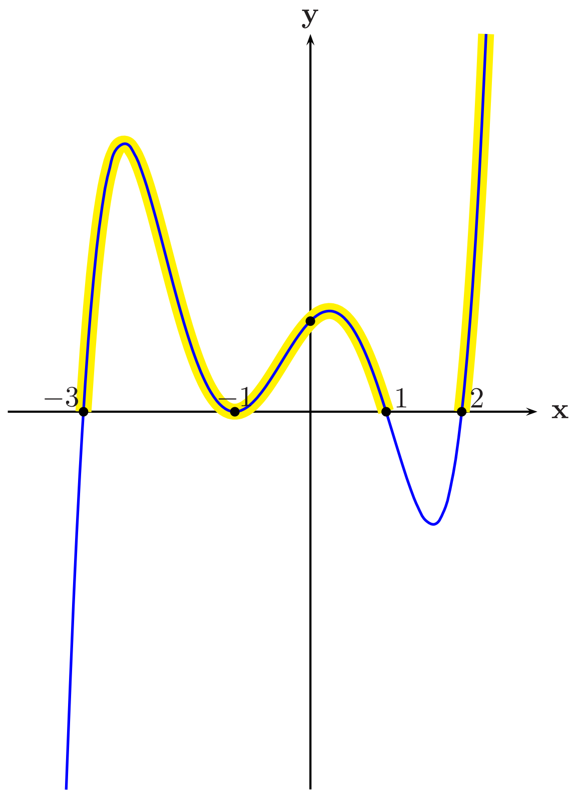
8. **Extra Credit:** Solve the inequality:

$$x^5 + 2x^4 - 6x^3 - 8x^2 + 5x + 6 \geq 0$$

Give your answer as a union of intervals.

Hint. The polynomial is the same as in Question 5.

Answer. Let $f(x) = x^5 + 2x^4 - 6x^3 - 8x^2 + 5x + 6$. Then $f(x) \geq 0$ means that the graph of $y = f(x)$ is above the x–axis. We have a qualitatively accurate graph of $y = f(x)$ from Question 5. Examining that graph we have:



Therefore the solution is the union of the following intervals:

$$[-3, 1] \cup [2, \infty)$$

□