Review for Chapter 5

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1. Can you figure out what is wrong in the following "proof"?

   Claim. All cars have the same color.

   Proof. There is only a finite number of cars in the world, let's say n. I will prove, using mathematical induction, that for any natural number $n \geq 1$, in any collection of n cars all cars have to have the same color.

   Base step: For $n = 1$ the proposition is obvious.

   Inductive step: Assume that the cars in any collections of n cars have the same color. I will prove that the cars in any collection of $n + 1$ cars have the same color. Indeed, let $\{c_1, c_2, \ldots, c_n, c_{n+1}\}$ be a collection of $n + 1$ cars. By the inductive hypothesis, the cars $c_1, c_2, \ldots, c_n$ have the same color so I need to show that the car c_{n+1} has the same color as well. But this is true because by the inductive hypothesis the cars $c_2, \ldots, c_n, c_{n+1}$ have the same color. □

2. Let $m \in \mathbb{N}$. Prove that for all integers $n > m$ we have:

$$\sum_{i=m+1}^{n} i = \frac{(n-m)(n+m+1)}{2}$$

3. Prove that for integers $n \geq 1$ we have:

$$\sum_{i=1}^{n} (3i - 1) = \frac{n(3n+1)}{2}$$

4. Prove by induction that for all integers $n \geq 1$ we have:

$$\sum_{i=1}^{n} (2i)^2 = \frac{2n(n+1)(2n+1)}{3}$$

5. Prove that for all integers $n \geq 1$ we have:

$$\sum_{i=1}^{n} (2i - 1)^2 = \frac{n(4n^2 - 1)}{3}$$

6. Prove that for integers $n \geq 1$ we have:

$$\sum_{i=1}^{n} 3^i = \frac{3^n - 1}{2}$$

7. Prove that for each $n \geq 1$ we have:

$$\sum_{i=0}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

8. Prove that for integers $n \geq 1$ we have:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

9. Prove that for integers $n \geq 1$ we have:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)(i+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

10. Show that for any $n \geq 0$ we have:

$$\prod_{i=0}^{n} (2i+1) = \frac{(2n+1)!}{2^n n!}$$

11. Prove that for all $n \in \mathbb{N}$ we have:

$$\cos(n\pi) = (-1)^n$$

12. If i is the imaginary unit (i.e. $i^2 = -1$) prove that:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

13. Prove that for natural numbers $n \geq 3$ we have $2n + 1 < n^2$.

14. Prove that for all positive integers we have:

$$\left(1 + \frac{3}{n}\right)^n \geq 1 + \frac{n}{3}$$

15. Prove that for integers $n \geq 4$, $2^n < n!$.

16. Prove that for integers $n \geq 5$, $n^2 < 2^n$.

17. Prove that for all $n \in \mathbb{N}$, 7 divides $n^7 - n$.

18. In the Country of Oz they have only 3–cent and 5–cent postage stamps. Prove that you can use these stamps to pay for any letter that costs 8 or more cents.

19. In Nevereverland chicken nuggets come in packages of 5 and 7. Prove that for $n \geq 24$ a Nevereverlander can combine packages to get a total of exactly n chicken nuggets.

20. Before the nugget company in Nevereverland decided to make nugget packages of 5 and 7 nuggets, they were making packages of 4 and 6 nuggets. Find all the possible number of nuggets that a Nevereverlander could get by combining packages back then.

21. Let g_n be the number of bitstrings of length n with no consecutive ones. Give a recursive formula for g_n and prove your answer.

22. The *mirror* of a bitstring s is a bitstring $m(s)$ that has the same length as s and differs in every bit from s. For example the mirror of 1010001 is 0101110. Give a recursive definition of $m(s)$ for a bitstring s.

23. Let $\Sigma = \{a, b, c\}$ be an alphabet.
    (a) Give a recursive definition of the set of palindromes in Σ^*.
    (b) Give a recursive definition for the number of palindromes of length n in Σ^*.
    (c) Find a non-recursive formula for the number of palindromes of length n in Σ^*.

24. On the set Σ^* of words from the alphabet $\Sigma = \{I, M, W\}$ define the flip $F(s)$ of a word s as follows:
    - $F(\lambda) = \lambda$, where λ is the empty word
    - For a word s, $F(sI) = F(s)I$, $F(sW) = F(s)M$, and $F(sM) = F(s)W$

    Call a word *flippant* if $F(s) = R(s)$, where $R(s)$ stands for the reverse of s. For example, MIW is a flippant word.
    (a) Give a recursive definition for the set of flippant words.
    (b) How many flippant words of length n are there? Give a formula and prove it.

25. The Fina Bocci company breeds worms for fishing. After each worm is two weeks old they cut off its tail which becomes a new worm. The tail grows back in a week, so once a worm becomes two weeks old it produces a new worm every week.
    (a) Assuming that no worm ever dies and that the company starts with one newly "born" worm, find a recursive formula for the the number of worms after n weeks.
    (b) Prove that the formula you found in part a) is correct.

26. Consider a $n \times 2$ checker board, and let t_n be the number of ways that we can completely tile the board using dominoes. Find a recursive formula for t_n.

27. The *Lucas numbers* are defined by the same recursive formula as the Fibonacci numbers but with different *initial conditions*, namely the nth Lucas number L_n is defined by:

$$L_n = \begin{cases} 2 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ L_{n-1} + L_{n-2} & \text{if } n \geq 2 \end{cases}$$

(a) If F_n is the nth Fibonacci number prove that for $n \geq 1$

$$F_{n-1} + F_{n+1} = L_n$$

(b) If φ and $\bar{\varphi}$ are the two solutions of the equation $x^2 = x + 1$, prove that for all $n \geq 0$:

$$L_n = \varphi^n + \bar{\varphi}^n$$

28. A Morse code is a word in the alphabet consisting of two letters, the dot "·" and the dash "–". The two letters have different length, the dot has length 1 while the dash has length 2.

(a) Give a recursive definition of the set of Morse codes M.

(b) Give a recursive definition of the length $l(s)$ of a Morse code s.

(c) Give a recursive formula for the number of Morse codes of length n. Prove this recursive formula.

29. For an integer n let c_n be the number of ways that n can be written as a sum of ones, twos, threes, or fours where the order that the summands are written is important. Find a recursive definition of c_n and prove your answer.

30. Alice and Bob play a game by taking turns removing up to 4 stones from a pile that initially has n stones. The person that removes the last stone wins the game. Alice plays always first. For which values of n does Alice have a winning strategy? For which values of n does Bob have a winning strategy? Prove your answer.

31. Chris and Deborah play a game by taking turns removing up to 4 stones from a pile that initially has n stones. The person that removes the last stone wins the game. Alice plays always first. For which values of n does Alice have a winning strategy? For which values of n does Bob have a winning strategy? Prove your answer.

32. Analyze a game that Alice and Bob, respectively Chris and Deborah, play according to the rules of the Question 30, Question 31 respectively, with the change that a player is allowed to remove m stones from the pile, for some positive integer m.