First Quiz for CSI35

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Directions: This quiz is due Monday 29, at 4:00 PM.

1. Prove that for all $n \in \mathbb{N}$, 5 divides $n^5 - n$.

2. Prove that for all $n \in \mathbb{N}$, 6 divides $n^3 - n$.

3. Alice and Bob play a game by taking turns removing 1, 2 or 3 stones from a pile that initially has n stones. The person that removes the last stone wins the game. Alice plays always first.

   (a) Prove by induction that if n is a multiple of 4 then Bob has a wining strategy.

   (b) Prove that if n is not a multiple of 4 then Alice has a wining strategy.

4. Chris and Dominique play a slightly different game. Again each player takes turns removing 1, 2 or 3 stones from a pile that initially has n stones but now, the person that removes the last stone loses the game. Chris plays always first. Analyze this game, that is, find the values of n for which Chris has a winning strategy and the values of n for which Dominique has a winning strategy. You should prove your result.

5. Prove that for all $n \in \mathbb{N}$,

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$$

6. Experiment with the first few values of $n \in \mathbb{N}$ to conjecture a formula for the value of

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n$$

   Then prove your conjecture using mathematical induction.

7. In a party with at least two people, every person shakes hands with the people they know. Any two given people will either not shake hands or they will shake hands exactly once. Show that there will always be at least one pair of people who shake the same number of hands.

8. For an integer n let g_n be the number of ways that n can be written as a sum of ones and twos, where the order that the summands are written is important. For example, $g(1) = 1$, $g(2) = 2$ since 2 can be written either as 2 or as $1+1$, and $g(3) = 3$ because 3 can be written as $1+1+1$ or as $1+2$ or as $2+1$.
   (a) Find a recursive definition of $g(n)$
   (b) Prove that this recursive definition is correct.

   For the next three questions f_n stands for the nth Fibonacci number.

9. Prove that for all $n \geq 1$ we have:
$$f_1^2 + f_2^2 + \cdots f_n^2 = f_n f_{n+1}$$

10. Prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$, for all positive integers n.

11. Let $\varphi = \dfrac{1+\sqrt{5}}{2}$, $\bar{\varphi} = \dfrac{1-\sqrt{5}}{2}$.
   (a) Prove that $\forall n \in \mathbb{N} \quad \varphi^n = f_{n-1} + f_n\varphi$ and $\bar{\varphi}^n = f_{n-1} + f_n\bar{\varphi}$.
   (b) Prove that
$$\forall n \in \mathbb{N} \quad f_n = \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}}$$

   For the next four questions recall that if $\Sigma = \{0, 1\}$ then the elements of Σ^*, i.e. the words on the alphabet Σ, are called *bit strings*.

12. How many bit strings of length n are there, where n is any natural number? Prove your answer.

13. For a bit string s, let $O(s)$ and $I(s)$ be number of zeroes and ones, respectively, that occur in s. So for example if $s = 01001$, then $O(s) = 3$ and $I(s) = 2$.
   (a) Give recursive definitions of $O(s)$ and $I(s)$.
   (b) If $l(s)$ stands for the length of s, prove that:
$$l(s) = O(s) + I(s)$$

14. The reverse of a string s is the string obtained by "reading s backwards", for example the reverse of the string "sub" is "bus". The reverse of a string s is denoted by s^R. Give a recursive definition of s^R, for bit strings s.

15. A *palindrome* is a string s such that $s^R = s$, in other words a string that reads the same when we read it backwards. For example the string "bob" is a palindrome.

(a) Give a recursive definition of the set Π of all bit strings that are palindromes.

(b) For a natural number n, how many bit string palindromes of length n are there? Prove your answer.

16. The set of *binary trees*, is recursively defined as follows:

- There is a a binary tree consisting of a single vertex r. The root of this tree is r.
- If T_1 and T_2 are two binary trees with roots r_1 and r_2 respectively, we can make a new binary tree by adding one new vertex r and two new edges connecting r to r_1 and r_2. The root of this new tree is r.
- All binary trees are constructed this way

For a binary tree, T let $v(T)$ and $e(T)$ denote the number of vertices and edges of T respectively.

(a) Give recursive definitions of $v(T)$ and $e(T)$.

(b) Prove that for all binary trees:

$$v(T) - e(T) = 1$$

17. **Extra Credit:** Can you explain the trick with the sum of numbers that I did last time?