First Quiz

The answers

1. Prove that for all $n \in \mathbb{N}$, 5 divides $n^5 - n$.

   Solution. We prove it by induction. For $n = 0$ the statement is

   $$5 \mid 0^5 - 0$$

   or equivalently

   $$5 \mid 0$$

   which is true. This completes the basic step. For the inductive step, we assume that it has been proven that $5 \mid n^5 - n$, that is we assume that there is a natural number k such that $n^5 - n = 5k$; and we will prove that $5 \mid (n+1)^5 - (n+1)$. Indeed, we have:

   $$\begin{aligned}(n+1)^5 - (n+1) &= n^5 + 5n^4 + 10n^2 + 5n + 1 - n - 1 \\ &= n^5 - n + 5(n^4 + 2n^2 + n) \\ &= 5k + 5(n^4 + 2n^2 + n) \\ &= 5(k + n^4 + 2n^2 + n)\end{aligned}$$

   So, $5 \mid (n+1)^5 - (n+1)$. □

2. Prove that for all $n \in \mathbb{N}$, 6 divides $n^3 - n$.

   Solution. The proof is by induction. For $n = 0$, the statement is true since 6 divides 0. Now we assume that the statement is proven for n, that is we assume that there is a $k \in \mathbb{N}$ such that $n^3 - n = 6k$; and we will prove that $6 \mid (n+1)^3 - (n+1)$. Indeed, we have:

   $$\begin{aligned}(n+1)^3 - (n+1) &= n^3 + 3n^2 + 3n + 1 - n - 1 \\ &= n^3 - n + 3(n^2 + n) \\ &= 6k + 3(2l) \\ &= 6k + 6l \\ &= 6(k + l)\end{aligned}$$

   where, to go from the second line to the third, we used the fact that $n^2 + n$ is even[1] and so can be written as $2l$ for some $l \in \mathbb{N}$. Therefore, $6 \mid (n+1)^3 - (n+1)$. □

3. Alice and Bob play a game by taking turns removing 1, 2 or 3 stones from a pile that initially has n stones. The person that removes the last stone wins the game. Alice plays always first.

   (a) Prove by induction that if n is a multiple of 4 then Bob has a wining strategy.

[1] Prove this!

Solution. We proceed by induction. Namely we will prove that for all $k \in \mathbb{N}$, if there are $4k$ stones in the initial pile then Bob has a winning strategy. For $k = 0$, it is Alice's turn to play and there are no stones left, so Bob wins. This completes the basic step. Assume now that the result has been proven for k, that is assume that it has been proven that when there are $4k$ stones in the initial pile, Bob has a winning strategy. To complete the inductive step we need to prove that when there are $4(k+1)$ stones in the initial pile, Bob has a winning strategy. Now, since $4(k+1) = 4k+4$, after Alice's first move, Bob can always leave $4k$ stones in the pile. Indeed, if Alice takes 1 stone, he takes 3; if she takes 2, he also takes 2; and if she takes 3 he takes 1. With this strategy, Bob can ensure that after his first move, in total 4 stones have been removed from the initial $4k+4$, thus leaving $4k$ stones in the pile. From that point on, Bob can follow the strategy of the second player for a game with $4k$ stones in the initial pile. Therefore Bob has a winning strategy, and this concludes the inductive step. □

(b) Prove that if n is not a multiple of 4 then Alice has a wining strategy.

Solution. If n is not a multiple of 4, then $n = 4k+1$, or $n = 4k+2$, or $n = 4k+3$ for some $k \in \mathbb{N}$. Then Alice in her first move can take 1, or 2, or 3 stones respectively, leaving a pile with $4k$ stones and Bob's turn to play. Now in effect she is the second player in a game that starts with $4k$ stones. Therefore according to part (a) she has a winning strategy. □

4. Chris and Dominique play a slightly different game. Again each player takes turns removing 1, 2 or 3 stones from a pile that initially has n stones but now, the person that removes the last stone loses the game. Chris plays always first. Analyze this game, that is, find the values of n for which Chris has a winning strategy and the values of n for which Dominique has a winning strategy[2]. You should prove your result.

Solution. Dominique has a winning strategy *if and only if* $n = 4k+1$ for some $k \in \mathbb{N}$.

We first prove the *if* part, by induction on k. For $k = 0$ we have a game with 1 stone and Chris's turn. He has to take the stone and therefore Dominique wins. This completes the basic step of the inductive proof. Now we assume that Dominique has a winning strategy when $n = 4k+1$ and we will prove that she has a winning strategy when $n = 4(k+1)+1$ as well. In a game with $4(k+1)+1 = 4k+5$, if Chris takes 1 Dominique takes 3, if he takes 2 she takes 2, and if he takes 3 she takes 1 thus ensuring that after the first two moves 4 stones have been removed in total, leaving a pile with $4k+1$ stones. She can then follow the strategy guaranteed by the inductive hypothesis. Therefore Dominique has a winning strategy, and this concludes the proof of the inductive step.

To prove the *only if* part, we need to prove that if the number of stones in the initial pile does not leave remainder 1 when divided by 4, then Chris has a winning strategy. So we need to prove that if $n = 4k$, or $n = 4k+2$, or $n = 4k+3$, then Chris has a

[2]Why these two possibilities cover all cases? In other words why one of the two players has to have a strategy, couldn't it be that no one has a strategy?

winning strategy. Indeed the first move of Chris is to take 3 or 1 or 2 stones respectively, leaving a pile with a number of stones that leave remainder 1 when divided by 4, and Dominique's turn to play. In effect he is then the second player in a game with $4k+1$ stones, and therefore he has a winning strategy. □

5. Prove that for all $n \in \mathbb{N}$,

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$$

Solution. For $n = 0$ the statement is

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^0 = \begin{pmatrix} a^0 & 0 & 0 \\ 0 & b^0 & 0 \\ 0 & 0 & c^0 \end{pmatrix}$$

or equivalently,

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is true. This completes the inductive step.

Now we assume that the statement has been proven for n and we will prove it for $n+1$, that is we will prove that

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^{n+1} = \begin{pmatrix} a^{n+1} & 0 & 0 \\ 0 & b^{n+1} & 0 \\ 0 & 0 & c^{n+1} \end{pmatrix}$$

We have

$$\begin{aligned} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^{n+1} &= \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^n \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \\ &= \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \\ &= \begin{pmatrix} a^{n+1} & 0 & 0 \\ 0 & b^{n+1} & 0 \\ 0 & 0 & c^{n+1} \end{pmatrix} \end{aligned}$$

This concludes the inductive step. □

6. Experiment with the first few values of $n \in \mathbb{N}$ to conjecture a formula for the value of

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n$$

Then prove your conjecture using mathematical induction.

Solution. We will prove that

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

We proceed by mathematical induction. For $n = 0$ the statement is:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^0 = \begin{pmatrix} 1 & 0 & \frac{0(0+1)}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or equivalently,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is true. This completes the basic step.

Now we assume that the statement is true for n and prove it for $n+1$. We have:

$$\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}^{n+1} = \begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}^{n} \begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$$

$$= \begin{pmatrix}1&n&\frac{n(n+1)}{2}\\0&1&n\\0&0&1\end{pmatrix}^{n} \begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$$

$$= \begin{pmatrix}1&n+1&1+n+\frac{n(n+1)}{2}\\0&1&n+1\\0&0&1\end{pmatrix}$$

$$= \begin{pmatrix}1&n+1&\frac{n(n+1)+2(n+1)}{2}\\0&1&n+1\\0&0&1\end{pmatrix}$$

$$= \begin{pmatrix}1&n+1&\frac{(n+1)(n+2)}{2}\\0&1&n+1\\0&0&1\end{pmatrix}$$

$$= \begin{pmatrix}1&n+1&\frac{(n+1)((n+1)+1)}{2}\\0&1&n+1\\0&0&1\end{pmatrix}$$

So we have shown that

$$\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}^{n+1} = \begin{pmatrix}1&n+1&\frac{(n+1)((n+1)+1)}{2}\\0&1&n+1\\0&0&1\end{pmatrix}$$

which completes the inductive step. $\square$

7. In a party with at least two people, every person shakes hands with the people they know. Any two given people will either not shake hands or they will shake hands exactly once. Show that there will always be at least one pair of people who shake the same number of hands.

   Solution. Let n be the number of people in the party. We will prove the statement using induction on n. When $n = 2$ there are only two people in the party and they either shake hands or they don't. In the first case each has shaken one hand while in the latter case each has shaken 0 hands; in any case they have shaken the same number of hands. This concludes the proof of the basic step.

   Now we assume that the statement is true for n and we will prove that it is also true for $n+1$. So consider a party with $n+1$ people. If there is a person that doesn't shake any hands, the remaining n people shake hands "among themselves". Therefore, by the inductive hypothesis, there is at least a pair of those n that have shaken the same number of hands. On the other hand, if there is no person that has shaken 0 hands, each of the $n+1$ people has shaken, 1, or 2, $\ldots$, or n hands. So we have $n+1$ people and n possibilities for how many hands each has shaken. Since there aren't enough possibilities for each person to have shaken a different number of hands it follows that at least two people have shaken the same number of hands. It follows that in any case, there are at least two people that have shaken the same number of hands, and this concludes the inductive step. □

8. For a positive integer n let g_n be the number of ways that n can be written as a sum of ones and twos, where the order that the summands are written is important. For example, $g(1) = 1$, $g(2) = 2$ since 2 can be written either as 2 or as $1 + 1$, and $g(3) = 3$ because 3 can be written as $1 + 1 + 1$ or as $1 + 2$ or as $2 + 1$.

   (a) Find a recursive definition of $g(n)$

   Solution.

$$g_n = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ g_{n-1} + g_{n-2} & \text{if } n \geq 3 \end{cases}$$

□

   (b) Prove that this recursive definition is correct.

   Solution. For $n = 1$ and $n = 2$ the definition is correct, as explained in the statement of the question.

   For a positive integer n let S_n be the set of all finite sequences of ones and twos that have sum n, by definition then $g_n = |S_n|$. Also, let $S_{n;1}$ be the subset of S_n consisting of sequences that start with 1, and $S_{n;2}$ the subset of S_n consisting of sequences starting with 2. Then we have

$$S_n = S_{n;1} \cup S_{n;2}, \quad \text{and} \quad S_{n;1} \cap S_{n;2} = \emptyset$$

   It follows that

$$|S_n| = |S_{n;1}| + |S_{n;2}|$$

   Since $g_n = |S_n|$, it suffices then to prove that for $n \geq 3$,

$$|S_{n;1}| = g_{n-1} \quad \text{and} \quad |S_{n;2}| = g_{n-2}$$

   Indeed we will prove that

$$|S_{n;1}| = |S_{n-1}| \quad \text{and} \quad |S_{n;2}| = |S_{n-2}|$$

   To prove that $|S_{n;1}| = |S_{n-1}|$ we will establish a bijection between these two sets.

We will define a function $f\colon S_{n;1} \to S_{n-1}$. By definition an element of $S_{n;1}$ has the form $1, s$ where s is a sequence of length $n-1$, and the sum of the terms of s is $n-1$. So we can define $f(1s) = s$. This function is one-to-one and onto. It is one-to-one because if $f(1s) = f(1s')$ then $s = s'$ and so $1s = 1s'$. It is also onto because for any $s \in S_{n-1}$, we have $1s \in S_{n;1}$ and $f(1s) = s$.

The proof that $|S_{n;2}| = |S_{n-2}|$ is entirely similar. $\square$

For the next three questions f_n stands for the nth Fibonacci number.

9. Prove that for all $n \geq 1$ we have:

$$f_1^2 + f_2^2 + \cdots f_n^2 = f_n f_{n+1}$$

Solution. We proceed by induction on n. For $n = 1$, the statement becomes: $f_1^2 = f_1 f_2$ which is true since $f_1 = f_2 = 1$. So we established the basic step.

Now for the inductive step we assume the statement has been proved for n and we will prove it for $n+1$. So we assume that it has been proved that

$$f_1^2 + f_2^2 + \cdots f_n^2 = f_n f_{n+1}$$

and will prove that

$$f_1^2 + f_2^2 + \cdots f_{n+1}^2 = f_{n+1} f_{n+2}$$

We have:

$$\begin{aligned} f_1^2 + f_2^2 + \cdots f_{n+1}^2 &= f_1^2 + f_2^2 + \cdots + f_n^2 + f_{n+1}^2 \\ &= f_n f_{n+1} + f_{n+1}^2 \\ &= f_{n+1}(f_n + f_{n+1}) \\ &= f_{n+1} f_{n+2} \end{aligned}$$

where to go from the the first line to the second we used the inductive hypothesis, and to go from the third to the last we used the recursive definition of the Fibonacci numbers. $\square$

10. Prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$, for all positive integers n.

Solution. Again we proceed by induction. So let $P(n)$ be the statement:

$$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$$

For $n = 1$ we have the statement becomes:

$$f_1 = f_2$$

which is true. So we've accomplished the basic step.

For the inductive step we assume that we have proved $P(n)$

$$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$$

and prove $P(n+1)$ i.e. we'll prove:

$$f_1 + f_3 + \cdots + f_{2n+1} = f_{2n+2}$$

We have

$$\begin{aligned} f_1 + f_3 + \cdots + f_{2n+1} &= f_1 + f_3 + \cdots + f_{2n-1} + f_{2n+1} \\ &= f_{2n} + f_{2n+1} \\ &= f_{2n+2} \end{aligned}$$

where to go from the first line to the second we used the inductive hypothesis and from the second to the third we used the recursive definition of the Fibonacci numbers. □

11. Let $\varphi = \dfrac{1+\sqrt{5}}{2}$, $\bar{\varphi} = \dfrac{1-\sqrt{5}}{2}$.

 (a) Prove that $\forall n \geq 1 \quad \varphi^n = f_{n-1} + f_n\varphi$ and $\bar{\varphi}^n = f_{n-1} + f_n\bar{\varphi}$.

 Solution. We first note that φ and $\bar{\varphi}$ are the solutions of the quadratic equation:

$$x^2 = x + 1 \tag{1}$$

 We will prove by induction that if x is a solution to Equation (1) then it satisfies

$$\forall n \geq 1, \quad x^n = f_{n-1} + f_n x$$

 For $n = 1$, the statement is exactly Equation (1), so the statement is true for $n = 1$. For the inductive step we assume that the statement has been proven for n and we'll prove it for $n+1$, i.e. we'll prove that:

$$x^{n+1} = f_n + f_{n+1}x$$

 We have:

$$\begin{aligned} x^{n+1} &= x^n \cdot x \\ &= (f_{n-1} + f_n x)x \\ &= f_{n-1}x + f_n x^2 \\ &= f_{n-1}x + f_n(x+1) \\ &= f_{n-1}x + f_n x + f_n \\ &= f_n + (f_{n-1} + f_n)x \\ &= f_n + f_{n+1}x \end{aligned}$$

 where to go from the first line to the second we used the inductive hypothesis, to go from the third to the fourth we used the fact that x satisfies Equation (1), and to go from the sixth to the seventh we used the recursive definition of the Fibonacci sequence. □

(b) Prove that

$$\forall n \in \mathbb{N} \quad f_n = \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}}$$

Solution. For $n = 0$ the statement is true.
For $n \geq 1$, from part (a) we know that

$$\varphi^n = f_{n-1} + f_n\varphi \text{ and } \bar{\varphi}^n = f_{n-1} + f_n\bar{\varphi}$$

Subtracting these two equations we get:

$$\begin{aligned} \varphi^n - \bar{\varphi}^n &= f_{n-1} + f_n\varphi - f_{n-1} - f_n\bar{\varphi} \\ &= (\varphi - \bar{\varphi})f_n \\ &= \sqrt{5} \cdot f_n \end{aligned}$$

which is equivalent to the statement we want to prove. □

For the next four questions recall that if $\Sigma = \{0, 1\}$ then the elements of Σ^*, i.e. the words on the alphabet Σ, are called *bit strings*.

12. How many bit strings of length n are there, where n is any natural number? Prove your answer.

Solutions. There are 2^n bitstrings of length n. We'll prove this by induction. For $n = 0$ the statement is true since there is only one string of length 0, namely the empty string.

Assume that there are 2^n bitstrings of length n. By definition the set of bitstrings of length $n + 1$ is the disjoint union of two sets: B_0 the set of bitstrings that end with 0 and B_1 the set of bitstrings that end with 1. Each of these sets are in bijection with the set of bitstrings of length n.[3] So, by the inductive hypothesis, the set of bitstrings of length $n + 1$ is

$$2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$$

□

13. For a bit string s, let $O(s)$ and $I(s)$ be number of zeroes and ones, respectively, that occur in s. So for example if $s = 01001$, then $O(s) = 3$ and $I(s) = 2$.

(a) Give recursive definitions of $O(s)$ and $I(s)$.

Solution. If λ is the empty string then we define:

$$\begin{aligned} O(\lambda) &= 0 \\ O(s0) &= O(s) + 1 \\ O(s1) &= O(s) \end{aligned}$$

[3] Why?

and

$$\begin{aligned} I(\lambda) &= 0 \\ I(s0) &= I(s) \\ I(s1) &= I(s) + 1 \end{aligned}$$

□

(b) If $l(s)$ stands for the length of s, prove that:

$$l(s) = O(s) + I(s)$$

Solution. We prove this by structural recursion. For the empty string we have:

$$\begin{aligned} O(\lambda) + I(\lambda) &= 0 + 0 \\ &= 0 \\ &= l(\lambda) \end{aligned}$$

so the statement is true for the empty string.
Assume that the statement is true for the bitstring s. Then we have:

$$\begin{aligned} l(s0) &= l(s) + 1 \\ &= l(s) + 0 + 1 \\ &= O(s) + I(s) + 0 + 1 \\ &= O(s) + 1 + I(s) + 0 \\ &= O(s0) + I(s0) \end{aligned}$$

and

$$\begin{aligned} l(s1) &= l(s) + 1 \\ &= l(s) + 0 + 1 \\ &= O(s) + I(s) + 0 + 1 \\ &= O(s) + 0 + I(s) + 1 \\ &= O(s1) + I(s1) \end{aligned}$$

So the statement has been proven for $s0$ and $s1$. Therefore by structural induction the statement has been proven for all bitstrings. □

14. The reverse of a string s is the string obtained by "reading s backwards", for example the reverse of the string "sub" is "bus". The reverse of a string s is denoted by s^R. Give a recursive definition of s^R, for bit strings s.

Solution.

$$\begin{aligned} \lambda^R &= \lambda \\ (s0)^R &= 0s^R \\ (s1)^R &= 1s^R \end{aligned}$$

□

15. A *palindrome* is a string s such that $s^R = s$, in other words a string that reads the same when we read it backwards. For example the string "bob" is a palindrome.

(a) Give a recursive definition of the set Π of all bit strings that are palindromes.

Solution. The set Π of all bitstrings that are palindromes are defined as follows:

- The empty string λ, and the strings 0 and 1 are in Π
- If s is in Π so are $0s0$ and $1s1$.
- A string s is in Π exactly when it can be constructed using the previous two rules.

□

(b) For a natural number n, how many bit string palindromes of length n are there? Prove your answer.

Solution. Let p_n be the number of bitstring palindromes of length n. We will prove that:

$$p_n = 2^{\lceil \frac{n}{2} \rceil}$$

We will prove this by strong induction. The statement is true for $n = 0$ since the empty string is the only sting in Π with length 0. It is also true for $n = 1$ since there are two strings in Π of length 1, namely 0 and 1.

Now we assume that the statement has been proven for all $k \leq n$ and we'll prove it for n. If s is a palindrome of length n, with $n \geq 3$, then it has the form $0s'0$ or $1s'1$ where s' is a palindrome of length $n-2$, and conversely, for any any palindrome s' of length $n-2$ then $0s'0$ and $1s'1$ are palindromes of length n. It follows that $p_n = 2p_{n-2}$. So:

$$\begin{aligned} p_n &= 2p_{n-2} \\ &= 2 \cdot 2^{\lceil \frac{n-2}{2} \rceil} \\ &= 2^{\lceil \frac{n-2}{2} \rceil + 1} \\ &= 2^{\lceil \frac{n-2}{2} + 1 \rceil} \\ &= 2^{\lceil \frac{n}{2} \rceil} \end{aligned}$$

where to go from the first line to the second we used the inductive hypothesis and for the remaining calculations basic properties of the ceiling function. □

16. The set of *binary trees*, is recursively defined as follows:

- There is a a binary tree consisting of a single vertex r. The root of this tree is r.
- If T_1 and T_2 are two binary trees with roots r_1 and r_2 respectively, we can make a new binary tree by adding one new vertex r and two new edges connecting r to r_1 and r_2. The root of this new tree is r.
- All binary trees are constructed this way

For a binary tree, T let $v(T)$ and $e(T)$ denote the number of vertices and edges of T respectively.

(a) Give recursive definitions of $v(T)$ and $e(T)$.

Solution. To simplify the description let's use the notation $\bullet$ for the binary tree that consists of a single vertex, and $\wedge(T_1, T_2)$ for the binary tree obtained by T_1 and T_2 by adding a new vertex and connecting it to the roots of T_1 and T_2. We have the following recursive definitions for $v(T)$ then:

- $v(\bullet) = 1$
- $v(\wedge(T_1, T_2)) = v(T_1) + v(T_2) + 1$

and for $e(T)$:

- $e(\bullet) = 0$
- $v(\wedge(T_1, T_2)) = e(T_1) + e(T_2) + 2$

□

(b) Prove that for all binary trees:

$$v(T) - e(T) = 1$$

Solution. We proceed by structural induction. The statement is true for $\bullet$ since

$$v(\bullet) - e(\bullet) = 1 - 0 = 1$$

Next we assume that the statement is true for T_1 and T_2 and we'll prove it for $\wedge(T_1, T_2)$. We have:

$$\begin{aligned} v(\wedge(T_1, T_2)) - e(\wedge(T_1, T_2)) &= v(T_1) + v(T_2) + 1 - (e(T_1) + e(T_2) + 2) \\ &= v(T_1) + v(T_2) + 1 - e(T_1) - e(T_2) - 2 \\ &= v(T_1) - e(T_1) + v(T_2) - e(T_2) - 1 \\ &= 1 + 1 - 1 \\ &= 1 \end{aligned}$$

where in the first line we used the recursive definitions of v and e, and to go from the third to the fourth we used the inductive hypothesis. □

17. **Extra Credit:** Can you explain the trick with the sum of numbers that I did last time?

Solution. Let $(a_i)_{i \in \mathbb{N}}$ the sequence created by the rules I explained. You had to choose the first two terms, say you chose $a_0 = m$ and $a_1 = n$. Then the sequence is defined recursively as follows:

$$a_k = \begin{cases} m & \text{if } k = 0 \\ n & \text{if } k = 1 \\ a_{k-1} + a_{k-2} & \text{if } k \geq 2 \end{cases}$$

The following table then illustrates the first ten values of the sequences as well as the sum of them:

k	a_k
0	m
1	n
2	$m+n$
3	$m+2n$
4	$2m+3n$
5	$3m+5n$
6	$5m+8n$
7	$8m+13n$
8	$13m+21n$
9	$21m+34n$
sum	$55m+88n$

From which it follows that the sum is 11 times a_6. During the trick I asked you what was the sixth term, ostensibly to verify that your calculations so far were correct. Once I knew a_6 it was easy to multiply it with 11 and find the sum. $\square$