Midterm exam

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1. Prove by induction that for all natural numbers $n \geq 5$ the following identity holds:

$$\sum_{i=5}^{n} i = \frac{(n-4)(n+5)}{2}$$

2. In the Land of Oz, they have only 5-cent and 7-cent stamps. Prove that you can use combinations of these stamps to pay for any letter that costs 24 or more cents.

3. Recall the recursive definition of the Fibonacci numbers f_n:

$$f_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f_{n-1} + f_{n-2} & \text{if } n \geq 2 \end{cases}$$

   Prove that for all natural numbers $n \geq 1$ we have:

$$f_{n+3} = 3f_n + 2f_{n-1}$$

4. Let Σ be a set of symbols. Give a recursive definition for the set Σ^* of strings formed from symbols in Σ.

5. Consider the alphabet $\Sigma = \{p, q\}$. The mirror $m(w)$ of a string $w \in \Sigma^*$ is the string we get by reading the reflection of w in a mirror. A recursive definition of the mirror of a string is the following:

   - $m(\lambda) = \lambda$, where λ is the empty string.
   - $m(wp) = qm(w)$
   - $m(wq) = pm(w)$

   A string $w \in \Sigma^*$ is called *self-mirror* if $m(w) = w$.

   (a) Give a recursive definition of the set $\mathcal{M}$ of self-mirror strings in the alphabet Σ.

   (b) Give a formula for the number of words in $\mathcal{M}$ that have length n.

   (c) Prove the formula you gave in part (b).

6. Give the definition of an *antisymmetric* relation on a set A.

7. The relation R on the set of real numbers $\mathbb{R}$ is defined as follows:

$$R = \{(x, y) \in \mathbb{R}^2 : x^2 = y^2\}$$

Prove that:

(a) R is reflexive.

(b) R is symmetric.

(c) R is transitive.

8. Let R_1 and R_2 be the relations on $\{1, 2, 3\}$ represented the digraphs G_1 and G_2 shown in Figure 1.

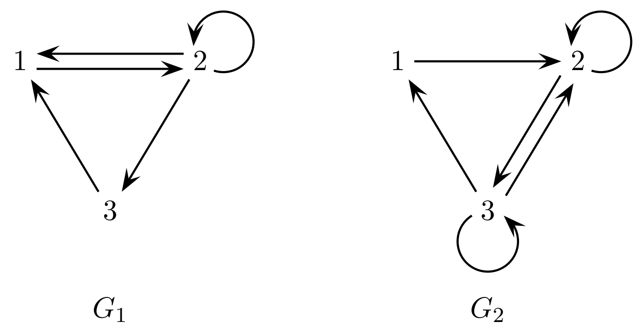

Figure 1: The digraphs of Question 8

(a) Find the matrices M_1 and M_2 representing the relations R_1 and R_2.

(b) Write down the relations R_1 and R_2 as sets of ordered pairs.

(c) Find the matrices corresponding to the relations $R_1 \circ R_2$ and $R_2 \circ R_1$.

(d) Write the relations $R_1 \circ R_2$ and $R_2 \circ R_1$ as sets of ordered pairs.

(e) Draw the digraphs representing the relations $R_1 \circ R_2$ and $R_2 \circ R_1$.

9. **Extra Credit:** Consider a $2 \times n$ checkerboard, and let t_n be the number of ways that we can completely tile the board using dominoes. For example as we see in Figure 2 we have $t_1 = 1$, $t_2 = 2$, and $t_3 = 3$. Find a recursive formula for t_n and prove that it is correct.

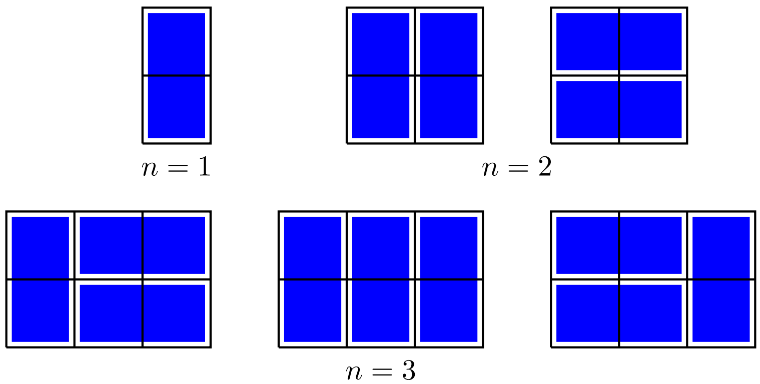

Figure 2: Tilling an $2 \times n$ board with dominoes for $n = 1, 2, 3$