The answers to Midterm Exam

Nikos Apostolakis

November 2, 2014

1. Prove by induction that for all natural numbers $n \geq 5$ the following identity holds:

$$\sum_{i=5}^{n} i = \frac{(n-4)(n+5)}{2}$$

Solution. Let $P(n)$ be the proposition:

$$P(n) \equiv \sum_{i=5}^{n} i = \frac{(n-4)(n+5)}{2}$$

We will prove by induction that $P(n)$ is true for all natural numbers $n \geq 5$. The basic step is to prove the $P(5)$. Indeed for $n = 5$ we have the proposition:

$$\sum_{i=5}^{5} i = \frac{(5-4)(5+5)}{2}$$

which is true since both sides are equal to 5. For the inductive step we need to prove that if for some natural number k, $P(k)$ is true then $P(k+1)$ is also true. So we assume that

$$\sum_{i=5}^{k} i = \frac{(k-4)(k+5)}{2}$$

and we will prove that

$$\sum_{i=5}^{k+1} i = \frac{(k+1-4)(k+1+5)}{2}$$

or equivalently:

$$\sum_{i=5}^{k+1} i = \frac{(k-3)(k+6)}{2}$$

We have:

$$\begin{aligned}
\sum_{i=5}^{k+1} i &= \sum_{i=5}^{k} i + (k+1) \\
&= \frac{(k-4)(k+5)}{2} + (k+1) \\
&= \frac{k^2 + k - 20 + 2k + 2}{2} \\
&= \frac{k^2 + 3k - 18}{2} \\
&= \frac{(k-3)(k+6)}{2}
\end{aligned}$$

where to go from the first line to the second we used the inductive hypothesis. So $P(k+1)$ has been proven, and the inductive step is completed. $\square$

2. In the Land of Oz, they have only 5-cent and 7-cent stamps. Prove that you can use combinations of these stamps to pay for any letter that costs 24 or more cents.

   Solution. We'll prove this by strong induction. Let $P(n)$ be the statement: "One can make a postage of n cents by using combinations of 5-cent and 7-cent stamps.".

   For the basic step we prove $P(24)$, $P(25)$, $P(26)$, $P(27)$, and $P(28)$. Indeed, $24 = 2 \times 5 + 2 \times 7$, $25 = 5 \times 5$, $26 = 1 \times 5 + 3 \times 7$, $27 = 4 \times 5 + 1 \times 7$, and $28 = 4 \times 7$.

   For the inductive step, we assume that $P(j)$ is true for all integers $24 \leq j \leq k$, where k is an integer with $k \geq 28$, and we will prove that $P(k+1)$ is true. Indeed since $28 \leq k$ then $24 \leq k - 4 < k$ so by the inductive hypothesis $P(k-4)$ is true so we can form postage of $k - 4$ cents using only 5-cent and 7-cent stamps. But then

$$k + 1 = (k - 4) + 5$$

   so we can form postage of $k+1$ cents by just adding one 5-cent stamp to the postage of $k - 4$ cents. This concludes the inductive step. $\square$

3. Recall the recursive definition of the Fibonacci numbers f_n:

$$f_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f_{n-1} + f_{n-2} & \text{if } n \geq 2 \end{cases}$$

   Prove that for all natural numbers $n \geq 1$ we have:

$$f_{n+3} = 3f_n + 2f_{n-1}$$

   Solution. We will use strong induction. Let $P(n) \equiv f_{n+3} = 3f_n + 2f_{n-1}$.

   For the basic step we prove $P(1)$, and $P(2)$. $P(1)$ is

$$f_4 = 3f_1 + 2f_0$$

which is true since $f_0 = 0$ and $f_1 = 1$, and $f_4 = 3$. $P(2)$ is

$$f_5 = 3f_2 + 2f_1$$

which is true since $f_5 = 5$, $f_1 = 1$, and $f_2 = 1$.

For the inductive step we assume that $P(j)$ is true for all integers $1 \leq j \leq k$, where k is an integer with $k \geq 2$ and we will prove that $P(k+1)$ is true. That is we will prove that

$$f_{k+4} = 3f_{k+1} + 2f_k$$

We have

$$\begin{aligned} f_{k+4} &= f_{k+3} + f_{k+2} \\ &= (3f_k + 2f_{k-1}) + (3f_{k-1} + 2f_{k-2}) \\ &= 3(f_k + f_{k-1}) + 2(f_{k-1} + f_{k-2}) \\ &= 3f_{k+1} + 2f_k \end{aligned}$$

where the first line follows from the recursive definition of the Fibonacci sequence, to go from the first line to the second we used the inductive hypothesis that $P(k)$ and $P(k-1)$ are true, and to go from the third line to the fourth we used the recursive definition of the Fibonacci numbers. □

4. Let Σ be a set of symbols. Give a recursive definition for the set Σ^* of strings formed from symbols in Σ.

   Solution. This is Definition 1, on page 349 of the seventh edition of the textbook. It's also Definition 2, on page 300 of the sixth edition. □

5. Consider the alphabet $\Sigma = \{p, q\}$. The mirror $m(w)$ of a string $w \in \Sigma^*$ is the string we get by reading the reflection of w in a mirror. A recursive definition of the mirror of a string is the following:

   - $m(\lambda) = \lambda$, where λ is the empty string.
   - $m(wp) = qm(w)$
   - $m(wq) = pm(w)$

   A string $w \in \Sigma^*$ is called *self-mirror* if $m(w) = w$.

   (a) Give a recursive definition of the set $\mathcal{M}$ of self-mirror strings in the alphabet Σ.

   (b) Give a formula for the number of words in $\mathcal{M}$ that have length n.

   (c) Prove the formula you gave in part (b).

   Solution. (a) The following is a recursive definition of $\mathcal{M}$:

   - The empty string λ is in $\mathcal{M}$.
   - If $s \in \mathcal{M}$ then $psq \in \mathcal{M}$.
   - If $s \in \mathcal{M}$ then $qsp \in \mathcal{M}$

(b) The following is a formula for s_n the number of self-mirror strings of length n.

$$s_n = \begin{cases} 2^k & \text{if } n = 2k \\ 0 & \text{if } n = 2k+1 \end{cases}$$

(c) For $n \in \mathbb{N}$ let $P(n)$ be the following proposition: "There are 2^n self-mirror strings of length $2n$ and no self-mirror strings of length $2n+1$.". We will prove this by induction.

For $n = 0$, $P(0)$ is true since there is one self-mirror string of length 0, namely the empty string, and there are no self-mirror strings of length 1 since neither p nor q are self-mirror.

For the inductive step we assume that $P(k)$ is true. To prove $P(k+1)$ we need to prove that there are 2^{k+1} self-mirror strings of length $2(k+1) = 2k+2$ and no self-mirror strings of length $2(k+1)+1 = 2k+3$.

Let s be a self-mirror string of length $2k+2$, then $s = ps\prime q$ or $s = qs\prime p$ for some self-mirror string $s\prime$ of length $2k$. Conversely if $s\prime$ is a self-mirror string of length $2k$ then $ps\prime q$ and $qs\prime p$ are self-mirror strings of length $2k+2$. It follows that

$$\begin{aligned} s_{2k+2} &= 2s_{2k} \\ &= 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$

where to go from the first line to the second we used the inductive hypothesis.

It remains to prove that there are no self-mirror strings of length $2k+3$. Arguing for contradiction, assume that s is a self-mirror string of length $2k+3$. Then $s = ps\prime q$ or $s = qs\prime p$ for some self-mirror string $s\prime$ of length $2k+1$. But since $P(k)$ is true, no such string $s\prime$ exists which is contradiction. Therefore there are no such strings s.

□

6. Give the definition of an *antisymmetric* relation on a set A.

Solution. This is the second part of Definition 4, on page 577 of the seventh edition of the textbook. It's also the second part of Definition 4, on page 523 of the sixth edition. □

7. The relation R on the set of real numbers $\mathbb{R}$ is defined as follows:

$$R = \{(x, y) \in \mathbb{R}^2 : x^2 = y^2\}$$

Prove that:

(a) R is reflexive.

Solution. Let $x \in \mathbb{R}$ then $x^2 = x^2$ and so $(x, x) \in R$. Thus R is reflexive. □

(b) R is symmetric.

Solution. Let $x, y \in \mathbb{R}$ so that $(x, y) \in R$, then $x^2 = y^2$ but then $y^2 = x^2$ and so $(y, x) \in R$. Thus R is symmetric. □

(c) R is transitive.

Solution. Let $x, y, z \in \mathbb{R}$ and assume that $(x, y) \in R$ and $(y, z) \in R$. Then $x^2 = y^2$ and $y^2 = z^2$. But then we have also that $x^2 = z^2$ and so $(x, z) \in R$. Thus R is transitive. $\square$

8. Let R_1 and R_2 be the relations on $\{1, 2, 3\}$ represented the digraphs G_1 and G_2 shown in Figure 1.

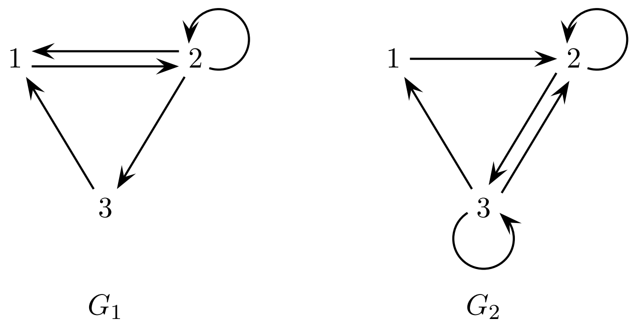


Figure 1: The digraphs of Question 8

(a) Find the matrices M_1 and M_2 representing the relations R_1 and R_2.

Solution.

$$M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$\square$

(b) Write down the relations R_1 and R_2 as sets of ordered pairs.

Solution.

$$R_1 = \{(1, 2), (2, 1), (2, 2), (2, 3), (3, 1)\}$$

and

$$R_2 = \{(1, 2), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

$\square$

(c) Find the matrices corresponding to the relations $R_1 \circ R_2$ and $R_2 \circ R_1$.

Solution. The matrix corresponding to $R_1 \circ R_2$ is

$$M_1 \odot M_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The matrix corresponding to $R_2 \circ R_1$ is

$$M_2 \odot M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$\square$

(d) Write the relations $R_1 \circ R_2$ and $R_2 \circ R_1$ as sets of ordered pairs.

Proof.

$$R_1 \circ R_2 = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,2)\}$$

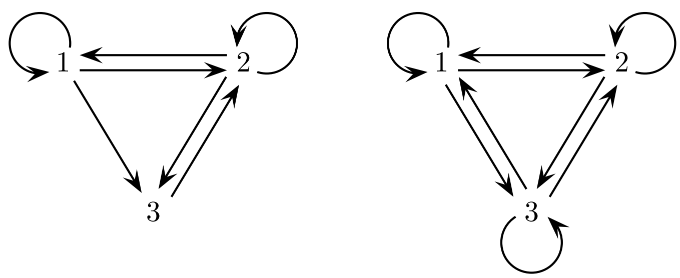
and

$$R_2 \circ R_1 = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$$

$\square$

(e) Draw the digraphs representing the relations $R_1 \circ R_2$ and $R_2 \circ R_1$.

Solution. See Figure 2. $\square$



$R_1 \circ R_2$ $R_2 \circ R_1$

Figure 2: The answer to Question 8 (e)

9. **Extra Credit:** Consider a $2 \times n$ checkerboard, and let t_n be the number of ways that we can completely tile the board using dominoes. For example as we see in Figure 3 we have $t_1 = 1$, $t_2 = 2$, and $t_3 = 3$. Find a recursive formula for t_n and prove that it is correct.

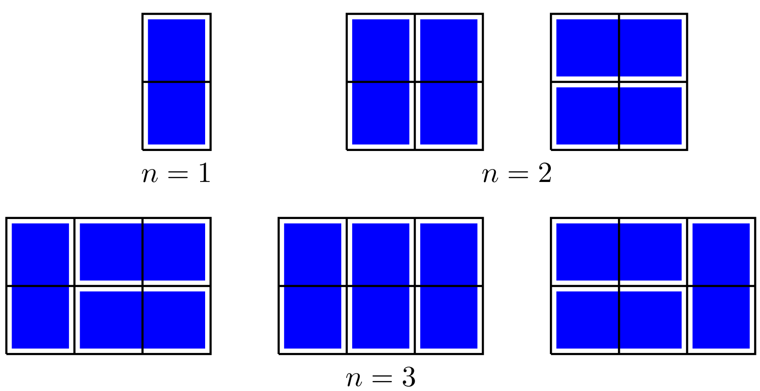


Figure 3: Tilling an $2 \times n$ board with dominoes for $n = 1, 2, 3$

Solution. We have the following recursive formula:

$$t_n = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ t_{n-1} + t_{n-2} & \text{if } n \geq 3 \end{cases}$$

As seen in Figure 3 the formula is correct for $n = 1$ and $n = 2$. For $n \geq 3$ notice that the tillings of the $2 \times n$ board fall into two disjoint categories, those that end with a vertical domino and those that end with two horizontal dominoes[1]. There are t_{n-1} tillings in the first category[2] and t_{n-2} in the second[3]. It follows that

$$t_n = t_{n-1} + t_{n-2}$$

$\square$

Followup Question

Here are some follow up questions. Their solution should be very similar to the questions in the exam.

1. Prove by induction that for all natural numbers n we have:

$$\sum_{i=5}^{n} i^2 = \frac{(n-4)(2n^2 + 11n + 45)}{6}$$

2. Prove that we can get any amount of 24 or more chicken nuggets by using only packages of 4 or 9 chicken nuggets.

3. **Extra Credit** What amounts of chicken nuggets can we get if we are only using packages that contain 3 or 6 chicken nuggets? Prove your answer.

4. If f_n is the n-th Fibonacci number, prove that for all $n \geq 1$ we have:

$$f_{n+5} = 8f_n + 5f_{n-1}$$

5. Consider the alphabet $\Sigma = \{p, o, q\}$. The mirror $m(w)$ of a string $w \in \Sigma^*$ is the string we get by reading the reflection of w in a mirror. A recursive definition of the mirror of a string is the following:

   - $m(\lambda) = \lambda$, where λ is the empty string.
   - $m(wo) = om(w)$
   - $m(wp) = qm(w)$
   - $m(wq) = pm(w)$

   A string $w \in \Sigma^*$ is called *self-mirror* if $m(w) = w$.

   (a) Give a recursive definition of the set $\mathcal{M}$ of self-mirror strings in the alphabet Σ.

   (b) Give a formula for the number of words in $\mathcal{M}$ that have length n.

   (c) Prove the formula you gave in part (b).

[1] For example the first tilling for $n = 3$ in Figure 3 falls into the the second category and the last two fall into the first.

[2] Why?

[3] Why?

6. The relation R on the set of real numbers $\mathbb{R}$ is defined as follows:

$$R = \{(x, y) \in \mathbb{R}^2 : \sin x = \sin y\}$$

Prove that R is an equivalence relation.

7. Consider the relations R, and S on the set $\{1, 2, 3, 4\}$ represented by the digraphs in Figure 4

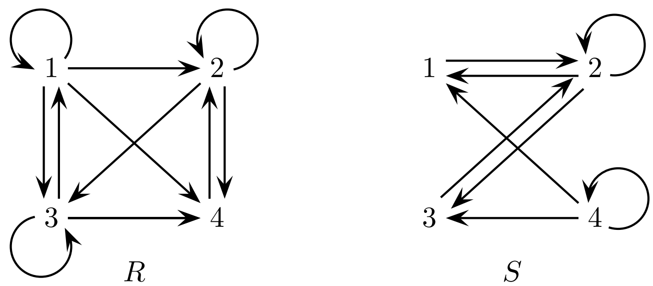


Figure 4: The digraphs of Question 6

   (a) Find the matrices M_S and M_R.
   (b) Use these matrices to compute the compositions $R \circ S$ and $S \circ R$.
   (c) Draw the digraphs that represent $R \circ S$ and $S \circ R$.

8. For $n \in \mathbb{N}$, let g_n be the number of bitstrings of length n that contain no consecutive ones. For example $g_0 = 1$, because the only bitstring of length 0 that does not contain two consecutive ones is the empty string, $g_1 = 2$ because we have the bitstrings 0 and 1, and $g_3 = 3$ because we have the bitstrings $00, 01, 10$. Give a recursive definition of g_n and prove that it is correct.

9. **Extra Credit** *This question is new, but you've seen similar ones before.* Julie and her partner invited n couples for dinner at their place. Afterwards Julie asked everybody (except herself of course) with how many people they had shook hands, and noticed that everybody gave a different number. Assuming that no one shook hands with their partner, prove that Julie's partner shook hands with exactly n people.