Final for CSI35

Nikos Apostolakis

December 9, 2014

Directions: This is the take home part of the final exam.

1. Let n be a positive integer. Prove that:

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right)=n+1$$

2. Prove, by induction or otherwise, that the sum of the cubes of any three consecutive natural numbers is divisible by 9.

3. Prove that for $n \geq 0$:

$$\cfrac{1}{1+\cfrac{1}{\ddots+\cfrac{1}{x}}}=\frac{f_n+f_{n+1}x}{f_{n+1}+f_{n+2}x}$$

where in the Left Hand Side there are $n+2$ fraction lines and f_n stands for the Fibonacci numbers.

4. For an integer n let $g(n)$ be the number of ways that n can be written as a sum of ones, twos, and threes, where the order that the summands are written is important. For example, $g(1)=1$, $g(2)=2$ since 2 can be written either as 2 or as $1+1$, and $g(3)=4$ because 3 can be written as $1+1+1$ or as $1+2$ or as $2+1$ or as 3.
   (a) Find a recursive definition of $g(n)$
   (b) Prove that this recursive definition is correct.

5. Let $\mathbb{N}^*$ be the set of positive integers. The relation $\sim$ on $\mathbb{N}^*$ is defined as follows:

$$m \sim n \iff \exists k \in \mathbb{N}^* \quad mn = k^2$$

   (a) Prove that $\sim$ is an equivalence relation.
   (b) Find the equivalence classes of 2, 4, and 6.

6. If the relation $\sim$ of Question 5 was defined on $\mathbb{N}$ instead of $\mathbb{N}^*$ would it still be an equivalence relation? Prove your answer.

7. There is a Christmas Carol called "The twelve days of Christmas":

   On the first day of Christmas,
   my true love sent to me
   A partridge in a pear tree.

   On the second day of Christmas,
   my true love sent to me
   Two turtle doves,
   And a partridge in a pear tree.

   On the third day of Christmas,
   my true love sent to me
   Three French hens,
   Two turtle doves,
   And a partridge in a pear tree.

   ...

   On the twelfth day of Christmas,
   my true love sent to me
   Twelve drummers drumming,
   Eleven pipers piping,
   Ten lords a-leaping,
   Nine ladies dancing,
   Eight maids a-milking,
   Seven swans a-swimming,
   Six geese a-laying,
   Five golden rings,
   Four calling birds,
   Three French hens,
   Two turtle doves,
   And a partridge in a pear tree!

   If the pattern continues for all natural numbers,

   (a) How many new gifts all together will her true love send her in the n–th day of Christmas?

   (b) How many gifts will she have accumulated in the n–th day of Christmas?

8. For $n = 0, 1, 2, 3, 4$ construct the Hasse diagram of the powerset of a set with n elements. What family of graphs you get? Will this pattern continue? Can you prove it?

9. Are the graphs in Figure 1 isomorphic?

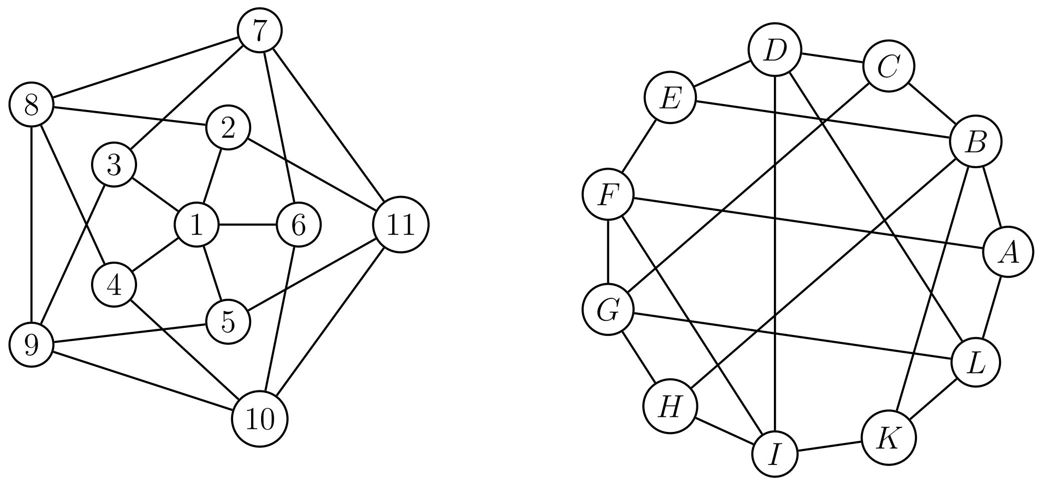

Figure 1: The two graphs of Question 9

10. For which values of n does the complete graph K_n have an Euler circuit?

11. For which values of n does the n-cube have an Euler circuit? How about a Hamiltonian circuit?

12. Give an example of a graph that has a Hamiltonian path but not a Hamiltonian circuit.

13. Find the shortest path from a to z in the weighted graph in Figure 2.

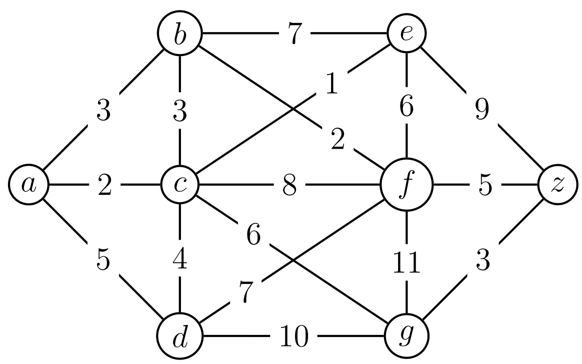

Figure 2: The weighted graph of Question 13

14. Prove that the n-cube Q_n is bipartite.

15. What's the chromatic number of the wheel W_n?

16. List all possible trees with six or less vertices.

17. Draw the game tree for the game of nim if the starting position consists of three piles with one, two and three stones respectively. Which player has a winning strategy?

18. How many children does the root of the game tree for nim have if the starting position consists of three piles with seven, five and three stones respectively?

19. For each of the graphs W_6, K_5, $K_{3,4}$, and Q_3 find a spanning tree using:

    (a) Depth-first search.

    (b) Breadth-first search.

20. Give a solution to the Eight-Queens puzzle.