Third Quiz for Math 31

The answers

1. State the Intermediate Value Theorem.

   Answer. It's on page 89, of the seventh edition of the textbook. □

2. Let $f(t) = t + t^{-1}$. A particle moves along a straight line with equation of motion $s = f(t)$, where s is measured in meters and t in seconds. Find the velocity of the particle for $t = \frac{1}{2}$, $t = 1$, and $t = 2$.

   Answer. The velocity of the particle is given by $v = f'(t)$. We first find a formula for $f'(t)$:

$$\begin{aligned}
f'(t) &= \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \\
&= \lim_{h\to 0} \frac{(t+h) + (t+h)^{-1} - (t + t^{-1})}{h} \\
&= \lim_{h\to 0} \frac{t + h + \dfrac{1}{t+h} - t - \dfrac{1}{t}}{h} \\
&= \lim_{h\to 0} \frac{h + \dfrac{1}{t+h} - \dfrac{1}{t}}{h} \\
&= \lim_{h\to 0} \left(1 + \frac{\dfrac{1}{t+h} - \dfrac{1}{t}}{h} \right) \\
&= 1 + \lim_{h\to 0} \frac{\dfrac{1}{t+h} - \dfrac{1}{t}}{h} \\
&= 1 + \lim_{h\to 0} \frac{\dfrac{t - (t+h)}{t(t+h)}}{h} \\
&= 1 + \lim_{h\to 0} \frac{\dfrac{-h}{t(t+h)}}{h} \\
&= 1 + \lim_{h\to 0} \frac{-1}{t(t+h)} \\
&= 1 - \frac{1}{t^2}
\end{aligned}$$

   Now we can calculate:

$$\begin{aligned}
v\left(\frac{1}{2}\right) &= 1 - \frac{1}{\left(\dfrac{1}{2}\right)^2} = 1 - 4 = -3 \\
v(1) &= 1 - \frac{1}{1^2} = 0 \\
v(2) &= 1 - \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4}
\end{aligned}$$

□

3. Let

$$f(x) = \sqrt{x-3}$$

(a) Use the definition of the derivative as a limit to find $f'(x)$.

Answer. We have:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h-3} - \sqrt{x-3}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h-3} - \sqrt{x-3}}{h} \frac{\sqrt{x+h-3} + \sqrt{x-3}}{\sqrt{x+h-3} + \sqrt{x-3}} \\
&= \lim_{h\to 0} \frac{(x+h-3) - (x-3)}{h\left(\sqrt{x+h-3} + \sqrt{x-3}\right)} \\
&= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h-3} + \sqrt{x-3}\right)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h-3} + \sqrt{x-3}} \\
&= \frac{1}{\sqrt{x-3} + \sqrt{x-3}} \\
&= \frac{1}{2\sqrt{x-3}}
\end{aligned}$$

□

(b) Find the tangent line to the graph of $y = f(x)$ at the point with coordinate $x = 12$.

Answer. The point on the graph of $y = f(x)$ with $x = 12$ has $y = \sqrt{12-3} = 3$. The slope of the tangent line at that point is $f'(12) = \frac{1}{2\sqrt{12-3}} = \frac{1}{6}$. Thus the tangent line has equation:

$$y - 3 = \frac{1}{6}(x - 12)$$

or equivalently

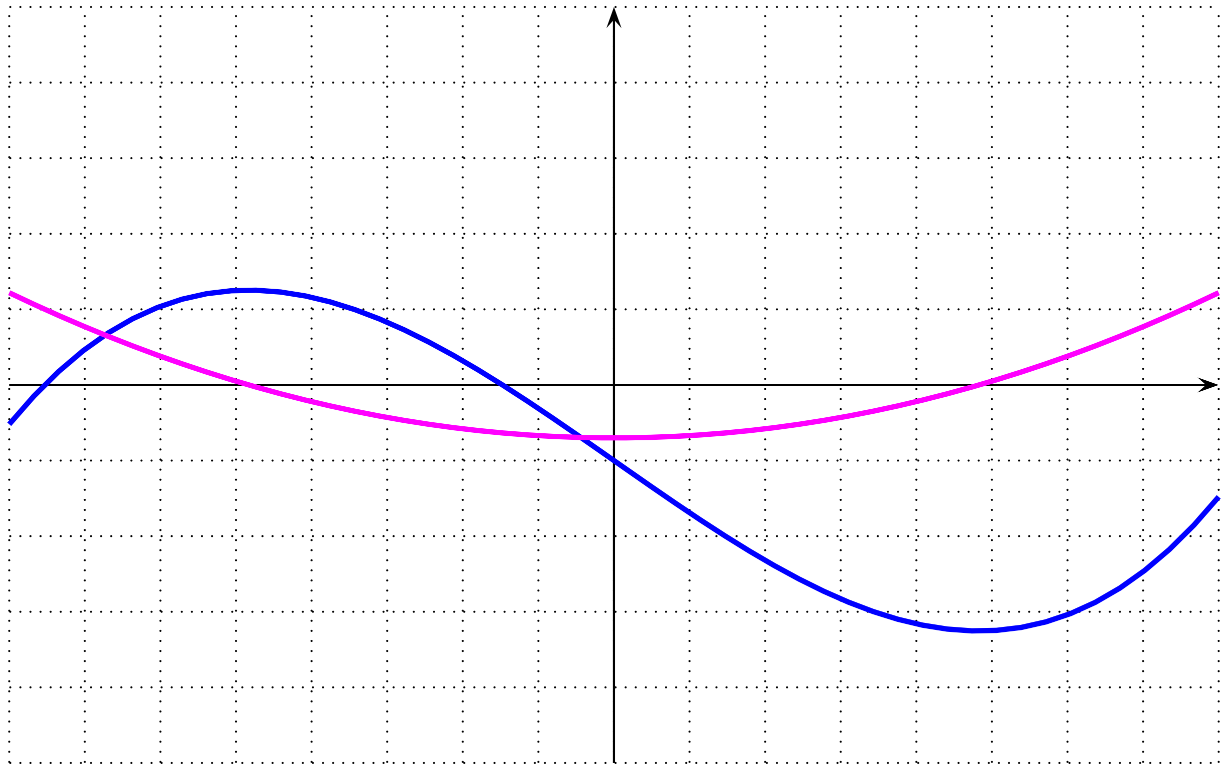
$$y - 3 = \frac{x}{6} - 2$$

which simplifies further to

$$y = \frac{x}{6} + 1$$

□

4. The graph of a function f and its derivative f' are shown below. Identify which graph is which.



Answer. The blue graph is the graph of f and the magenta graph the graph of f'. The derivative is the slope of the tangent line. So when the tangent line of the graph of f is horizontal the graph of f' should intersect the x-axis. □