Quiz 2
The answers

1. Does the limit $\lim_{x\to\infty} \sin x$ exist?

   Answer. This limit does not exist.

   First Justification: The reason is that $f(x) = \sin x$ is a periodic function with period 2π, so that as x gets larger and larger the values $f(x)$ do not get closer and closer to any given value, instead they oscillate between 1 and -1.

   Second Justification: Using the substitution $u = \frac{1}{x}$ we have that as $x \to \infty$, $u \to 0$ and $\sin x = \sin \frac{1}{u}$. So,

   $$\lim_{x\to\infty} \sin x = \lim_{u\to 0} \sin \frac{1}{u}$$

   We know that the limit in the Right Hand Side of the above equation does not exist, so the limit in the Left Hand Side doesn't exist either. □

2. Find the limit

   $$\lim_{x\to\infty} \frac{\sin x}{x}$$

   Answer. Using the squeeze theorem. We know that for all real numbers x,

   $$-1 \le \sin x \le 1$$

   It follows that for $x > 0$ we have that

   $$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x} \tag{1}$$

   Now

   $$\lim_{x\to\infty}\left(-\frac{1}{x}\right) = 0 \qquad \text{and} \qquad \lim_{x\to\infty} \frac{1}{x} = 0 \tag{2}$$

   According to the Squeeze theorem, (1) and (2) imply that

   $$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

   Second solution: Using the substitution $u = \frac{1}{x}$ we have that as $x \to \infty$, $u \to 0^+$ and so:

   $$\lim_{x\to\infty} \frac{\sin x}{x} = \lim_{u\to 0^+} \frac{1}{u} \sin u$$

   Now we saw in class that

   $$\lim_{u\to 0^+} \frac{1}{u} \sin u = 0$$

   Therefore:

   $$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

   □

3. Explain *in detail* why the function

$$f(x) = \frac{(2+x)^3 - 7}{1+x^2} - \sqrt{x^2+3} - \sin(\cos(3x))$$

is continuous on $(-\infty, \infty)$.

Answer. Let

$$f_1(x) = \frac{(2+x)^3 - 7}{1+x^2}$$
$$f_2(x) = \sqrt{x^2+3}$$
$$f_3(x) = \sin(\cos(3x))$$

Then

- f_1 is a rational function with domain $(-\infty, \infty)$ since its denominator $1+x^2$ is never zero. Therefore f_1 is continuous on $(-\infty, \infty)$.
- f_2 a root function with domain $(-\infty, \infty)$ since x^2+3 is positive for all real numbers.
- f_3 is the composition of three functions continuous on $(-\infty, \infty)$ and therefore is continuous on $(-\infty, \infty)$.

Now $f(x) = f_1(x) - f_2(x) - f_3(x)$ and therefore is continuous on $(-\infty, \infty)$ as a sum/difference of continuous functions. □

4. Find the points that each of the following functions is discontinuous and identify the nature of the discontinuity:

(a) $f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Answer. Since

$$\frac{|x|}{x} = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

we have:

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Now if $a \neq 0$, then $f(x)$ is constant near a (1 if $a > 0$ and -1 if $a < 0$) and therefore f is continuous at a. On the other hand if $a = 0$ the limit $\lim_{x\to 0}$ doesn't exist because the two side limits are different, namely $\lim_{a\to 0^-} = -1$ while $\lim_{x\to 0^+} = 1$. Therefore f has a *jump discontinuity* at 0. □

(b) $g(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 3 & \text{if } x = 0 \end{cases}$

Answer. For $a \neq 0$, near a, g is the quotient of a continuous function ($f(x) = \sin x$ and a non-zero continuous function ($h(x) = x$) and therefore g is continuous at a. On the other hand, at $a = 0$ we have,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1$$

while

$$f(0) = 3$$

Since the limit $\lim_{x\to 0} f(x)$ exists and is different than $f(0)$, g has a *removable discontinuity* at 0. $\square$

(c) $g(x) = \begin{cases} \dfrac{3}{(x-5)^2} & \text{if } x \neq 5 \\ 5 & \text{if } x = 5 \end{cases}$

Answer. For $a \neq 5$, g near a is equal to a rational function and so continuous at a. On the other hand, at $a = 5$, we have $\lim_{x\to 5} g(x) = +\infty$ so g has an *infinite discontinuity* at 5. $\square$

5. Find the real number a so that the function defined by

$$f(x) = \begin{cases} 2x - a & \text{if } -\infty \leq x \leq \pi \\ \sin x & \text{if } \pi < x < \infty \end{cases}$$

is continuous on $\mathbb{R}$.

Answer. The function is continuous at $(-\infty, \pi)$ being a linear function, and on (π, ∞) being a trigonometric function. So in order for f to be continuous on $\mathbb{R}$ it needs to be continuous at π, so we need the limit $\lim_{x\to\pi}$ to exist and

$$\lim_{x\to\pi} f(x) = f(\pi) \tag{3}$$

Now,

$$\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^-} (2x - a) = 2\pi - a$$

and

$$\lim_{x\to\pi^+} f(x) = \lim_{x\to\pi^+} \sin x = 0$$

so in order for $\lim_{x\to\pi} f(x)$ to exist we need $2\pi - a = 0$, or equivalently, $a = 2\pi$. When $a = 2\pi$ we have $f(\pi) = 0$ and $\lim_{x\to\pi} f(x) = 0$ so Equation (3) is satisfied and therefore f is continuous at $\mathbb{R}$. $\square$

6. Give an example of a function that

(a) has a jump discontinuity at $x = -5$.

Answer. We need a function f with

$$\lim_{x\to -5^-} \neq \lim_{x\to -5^+}$$

One such example is

$$f(x) = \begin{cases} 2x & \text{if } x \leq -5 \\ 3x & \text{if } x \geq -5 \end{cases}$$

$\square$

(b) has a removable discontinuity at $x = 0$.

Answer. We need a function f with

$$\lim_{x\to 0} f(x) \neq f(0)$$

One such example is

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 42 & \text{if } x = 0 \end{cases}$$

□

(c) has an infinite discontinuity at $x = 3$.

Answer. We need a function f with an infinite limit as $x \to 3$. One such example is

$$f(x) = \frac{1}{x-3}$$

□

(d) is continuous everywhere except at $x = 0$ and the discontinuity is not jump, removable or infinite.

Answer. Such a function is

$$f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 42 & \text{if } x = 0 \end{cases}$$

Since the limit $\lim_{x\to 0} f(x)$ does not exist the function is discontinuous at 0 and the discontinuity is not removable. Since the side limits $\lim_{x\to 0^\pm} f(x)$ don't exist the discontinuity is not jump, and since $\lim_{x\to 0} f(x) \neq \pm\infty$ the discontinuity is not infinite either. □

7. Prove that the equation $2^x = x^2$ has a solution in the interval $[-1, 0]$. Use a computer or a calculator to approximate that solution to the second decimal place.

Answer. Consider the function $f(x) = 2^x - x^2$, we need to show that for some c in the interval $[-1, 0]$, $f(c) = 0$. We will use the Intermediate Value Theorem.

The function f is continuous on $\mathbb{R}$ and therefore at the interval $[-1, 0]$. We have $f(-1) = 2^{-1} - (-1)^2 = -\frac{1}{2}$ and $f(0) = 1$. Since 0 is between these two values it follows by the I.V.T. that for some c in $(-1, 0)$, $f(c) = 0$.

To approximate solution to the second decimal place we keep getting smaller intervals that contain a solution by bisecting the interval at every step and selecting the appropriate half. We continue until the two endpoints of the interval agree to the second decimal place. We get Table 7. So, $c \approx -0.76$. □

8. Use the fact that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

to evaluate the following limits

(a) $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{5x}$

Answer. Use the substitution $u = 5x$, if $x \to 0$ then $u \to 0$:

$$\lim_{x\to 0} \frac{\sin 5x}{5x} = \lim_{u\to 0} \frac{\sin u}{u} = 1$$

□

x	$2^x - x^2$
-1	$-$
0	$+$
-0.5	$+$
-0.75	$+$
-0.875	$-$
-0.8125	$-$
-0.78125	$-$
-0.765625	$+$
-0.7734375	$-$
-0.76953125	$-$
-0.767578125	$-$
-0.7666015625	$+$

Table 1: Approximating a root of $2^x = x^2$

(b) $\lim_{x \to 0} \frac{\sin 3x}{x}$

Answer.

$$\begin{aligned} \lim_{x \to 0} \frac{\sin 3x}{x} &= \lim_{x \to 0} 3 \frac{\sin 3x}{3x} \\ &= 3 \lim_{x \to 0} \frac{\sin 3x}{3x} \\ &= 3 \lim_{x \to 0} \frac{\sin 3x}{3x} \\ &= 3 \cdot 1 \\ &= 3 \end{aligned}$$

□

(c) $\lim_{x \to 0} \frac{\sin 2x}{3x}$

Answer.

$$\begin{aligned} \lim_{x \to 0} \frac{\sin 2x}{3x} &= \lim_{x \to 0} \frac{2}{3} \frac{\sin 2x}{2x} \\ &= \frac{2}{3} \lim_{x \to 0} \frac{\sin 2x}{2x} \\ &= \frac{2}{3} \lim_{x \to 0} \frac{\sin 2x}{2x} \\ &= \frac{2}{3} \cdot 1 \\ &= \frac{2}{3} \end{aligned}$$

□

(d) $\lim_{x \to 0} \frac{\cos x - 1}{x}$

Answer.

$$\begin{aligned}
\lim_{x \to 0} \frac{\cos x - 1}{x} &= \lim_{x \to 0} \frac{\cos x - 1}{x} \frac{\cos x + 1}{x} \\
&= \lim_{x \to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} \\
&= \lim_{x \to 0} \frac{-\sin^2 x}{x(\cos x + 1)} \\
&= -\lim_{x \to 0} \frac{\sin x}{x} \lim_{x \to 0} \frac{\sin x}{\cos x + 1} \\
&= -1 \cdot \frac{0}{1+1} \\
&= 0
\end{aligned}$$

□