First Quiz for Math 31

The answers

1. Calculate the following limits. If you think a certain limit doesn't exist state so and explain why.

   (a) $\lim_{x\to 2} \dfrac{x^2 - x - 2}{x^2 + x - 6}$.

   Answer. The limit of both the numerator and the denominator as $x \to 2$ is 0 so we have a 0/0 indeterminate form. We have that for $x \neq 2$,

$$\begin{aligned} \frac{x^2 - x - 2}{x^2 + x - 6} &= \frac{(x-2)(x+1)}{(x-2)(x+3)} \\ &= \frac{x+1}{x+3} \end{aligned}$$

   So

$$\begin{aligned} \lim_{x\to 2} \frac{x^2 - x - 2}{x^2 + x - 6} &= \lim_{x\to 2} \frac{x+1}{x+3} \\ &= \frac{2+1}{2+3} \\ &= \frac{3}{5} \end{aligned}$$

   □

   (b) $\lim_{x\to -4} \dfrac{|x+4|}{x+4}$.

   Answer. We have

$$|x+4| = \begin{cases} x+4 & \text{if } x+4 \geq 0 \\ -(x+4) & \text{if } x+4 < 0 \end{cases}$$

   or equivalently:

$$|x+4| = \begin{cases} x+4 & \text{if } x \geq -4 \\ -(x+4) & \text{if } x < -4 \end{cases}$$

   Therefore:

$$\frac{|x+4|}{x+4} = \begin{cases} 1 & \text{if } x+4 > 0 \\ -1 & \text{if } x+4 < 0 \end{cases}$$

   It follows that

$$\lim_{x\to -4^-} \frac{|x+4|}{x+4} = \lim_{x\to -4^-} (-1) = -1$$

   while

$$\lim_{x\to -4^+} \frac{|x+4|}{x+4} = \lim_{x\to -4^+} 1 = 1$$

   Since the left- and right-hand limits are different it follows that the limit **does not exist.** □

   (c) $\lim_{t\to 0} \dfrac{\sqrt{t^2+9} - 3}{t^2}$

Answer. Again we have a 0/0 indeterminate form. Since there is a radical expression in the numerator we'll multiply numerator and denominator with its conjugate radical expression:

$$\begin{aligned}\frac{\sqrt{t^2+9}-3}{t^2} &= \frac{\sqrt{t^2+9}-3}{t^2}\,\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} \\ &= \frac{(t^2+9)-9}{t^2(\sqrt{t^2+9}+3)} \\ &= \frac{t^2}{t^2(\sqrt{t^2+9}+3)} \\ &= \frac{1}{\sqrt{t^2+9}+3}\end{aligned}$$

So,

$$\begin{aligned}\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2} &= \lim_{t\to 0}\frac{1}{\sqrt{t^2+9}+3} \\ &= \frac{1}{\sqrt{0^2+9}+3} \\ &= \frac{1}{6}\end{aligned}$$

□

(d) $\displaystyle\lim_{x\to\infty}\frac{5x^4-3x^2+2x-1}{-3x^3+7x^2-8}$

Answer. In this case we have a ∞/∞ indeterminate form. We have:

$$\begin{aligned}\lim_{x\to\infty}\frac{5x^4-3x^2+2x-1}{-3x^3+7x^2-8} &= \lim_{x\to\infty}\frac{x^4\left(5-\frac{3}{x^2}+\frac{2}{x^3}-\frac{1}{x^4}\right)}{x^3\left(-3+\frac{7}{x}-\frac{8}{x^3}\right)} \\ &= \lim_{x\to\infty} x\frac{5-\frac{3}{x^2}+\frac{2}{x^3}-\frac{1}{x^4}}{-3+\frac{7}{x}-\frac{8}{x^3}} \\ &= \infty\cdot\frac{5-0+0-0}{-3+0+0} \\ &= \infty\cdot\left(-\frac{5}{3}\right) \\ &= -\infty\end{aligned}$$

□

2. Let

$$f(x)=\begin{cases} ax^2-3x+4 & \text{if } x\le 2 \\ x+3a & \text{if } x>2\end{cases}$$

Find the real number a so that $\displaystyle\lim_{x\to 2} f(x)$ exists.

Answer. In order for $\lim_{x\to 2} f(x)$ to exist we need the left- and right-hand limits to exist and be equal. So we need:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

Now:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (ax^2 - 3x + 4) = 4a - 2$$

and

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x + 3a) = 3a + 2$$

Thus in order for the limit to exist we need

$$4a - 2 = 3a + 2$$

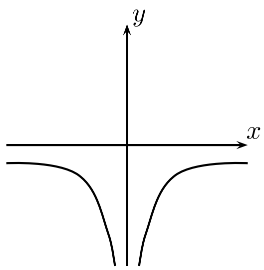
or equivalently,

$$a = 4$$

□

3. By examining the graphs calculate the required limits. If you think that a certain limit doesn't exist state so.

a)

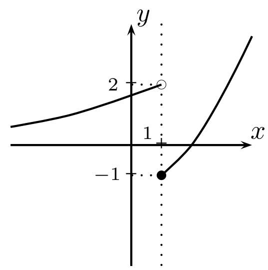


$$\lim_{x\to 0^+} f(x) = -\infty$$

$$\lim_{x\to 0} f(x) = -\infty$$

$$\lim_{x\to 0^-} f(x) = -\infty$$

b)

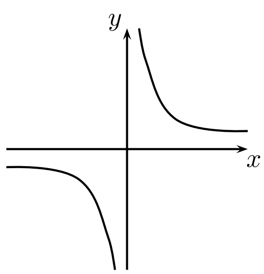


$$\lim_{x\to 1^+} f(x) = -1$$

$$\lim_{x\to 1} f(x) = \text{D.N.E.}$$

$$\lim_{x\to 1^-} f(x) = 2$$

c)

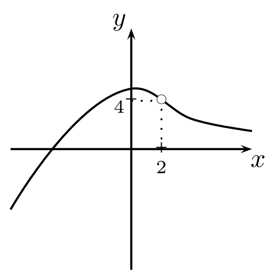


$$\lim_{x\to 0^+} f(x) = +\infty$$

$$\lim_{x\to 0} f(x) = \text{D.N.E.}$$

$$\lim_{x\to 0^-} f(x) = -\infty$$

c)



$$\lim_{x\to 2^+} f(x) = 4$$

$$\lim_{x\to 2} f(x) = 4$$

$$\lim_{x\to 2^-} f(x) = 4$$