Second Exam

THE ANSWERS

1. Prove that the equation
$$2x^3 + 9x^2 + 42x - 5 = 0$$
has exactly one real solution.

Answer. Let $f(x) = 2x^3 + 9x^2 + 42x - 5$. Then f, being a polynomial map, is continuous and differential on $\mathbb{R}$. Now $f(0) = -5$ and $f(1) = 48$, and since 0 is between -1 and 48 it follows from the Intermediate Value Theorem that for some c in $(0, 1)$ we have $f(c) = 0$, i.e. that c is a solution to the given equation.

On the other hand we have
$$f'(x) = 6x^2 + 18x + 42 = 6(x^2 + 3x + 7)$$
Now $x^2 + 3x + 7$ is a quadratic polynomial with negative discriminant (indeed, $D = 9 - 4 \cdot 7 = -19$), so it has no real zeros. It follows that $f'(x) \neq 0$ for all real numbers x and therefore f is a one-to-one function. Thus $f(x) = 0$ can not have two different solutions.

So, $f(x) = 0$ has exactly one solution. □

2. Let $f(x) = 3x^4 + 4x^3 - 12x^2 - 10$.

(a) Find the (absolute) extremum values of f in the interval $[-3, 2]$.

Answer. The extrema will occur at the endpoints or at the critical points of f. Since f is differentiable everywhere the only critical points of f occur when the derivative f' is zero. We have:
$$f'(x) = 12x^3 + 12x^2 - 24x$$
So
$$\begin{aligned} f'(x) = 0 &\iff 12x^3 + 12x^2 - 24x = 0 \\ &\iff 12x(x^2 + x - 2) = 0 \\ &\iff 12x(x + 2)(x - 1) = 0 \\ &\iff x = 0, \text{ or } x = -2, \text{ or } x = 1 \end{aligned}$$
We have the following table of the values of f at the endpoints and critical points:

x	$f(x)$
-3	17
-2	-42
1	-15
0	-10
2	22

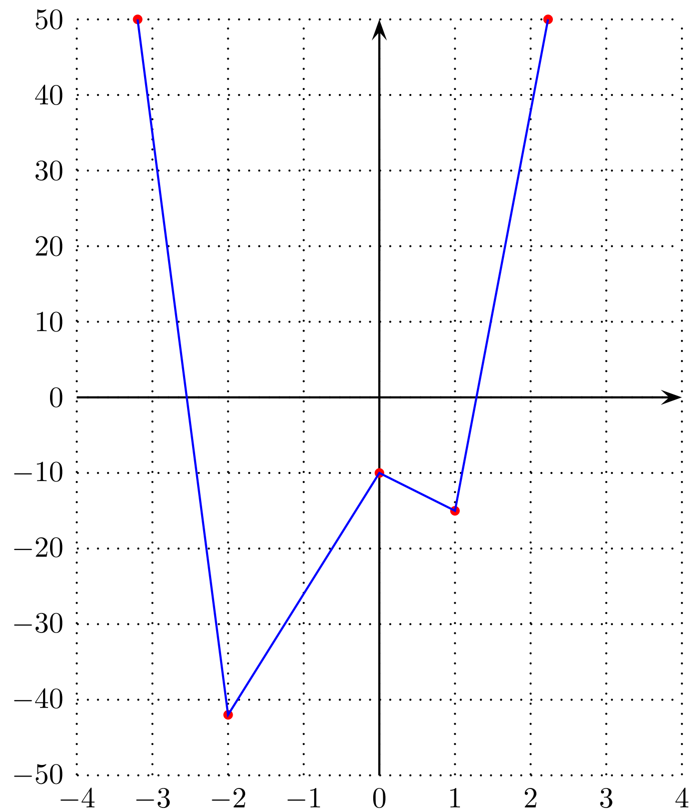
So the absolute minimum value is -42 and it occurs at $x = -2$ while the absolute maximum value is 22 and it occurs at $x = 2$. □

(b) How many real solutions does the equation $f(x) = 0$ have?

Answer. We will sketch a "stick" graph of $y = f(x)$. For this we need to know the sign of the first derivative. We already know the critical points of f from the previous part, and we can construct the following table of signs:

x	$-\infty$		-2		0		1		∞
$f'(x)$		$-$	0	$+$	0	$-$	0	$+$	
$f(x)$		↘	-42	↗	-10	↘	-15	↗ 22	

So we have the following very rough "stick" graph for f:



From the graph we see that the equation $f(x) = 0$ has exactly two real solutions. □

3. Sketch a graph of the function

$$f(x) = |x^3 - 2x^2 + x|$$

The graph should correctly indicate x and y intercepts, local extrema, points of inflection, the intervals where f is increasing or decreasing, and the intervals where f is concave upwards or downwards.

Answer. We will first graph the function $g(x) = x^3 - 2x^2 + x$, and then from this graph we will deduce the graph of f. We first calculate the x-intercepts:

$$\begin{aligned} g(x) = 0 &\iff x^3 - 2x^2 + x = 0 \\ &\iff x(x^2 - 2x + 1) = 0 \\ &\iff x(x-1)^2 = 0 \\ &\iff x = 0, \text{ or } x = 1 \end{aligned}$$

The y-intercept is the origin $(0,0)$.

We next examine the end behavior of $g(x)$:

$$\lim_{x\to-\infty} g(x) = -\infty, \qquad \lim_{x\to\infty} g(x) = \infty,$$

Next we calculate the critical points of g. Since g is differentiable everywhere the only critical points will be the zeros of the fist derivative. We have:

$$g'(x) = 3x^2 - 4x + 1$$

So:

$$\begin{aligned} g'(x) = 0 &\iff 3x^2 - 4x + 1 = 0 \\ &\iff x = \frac{4 \pm \sqrt{4}}{6} \\ &\iff x = 1, \text{ or } x = \frac{1}{3} \end{aligned}$$

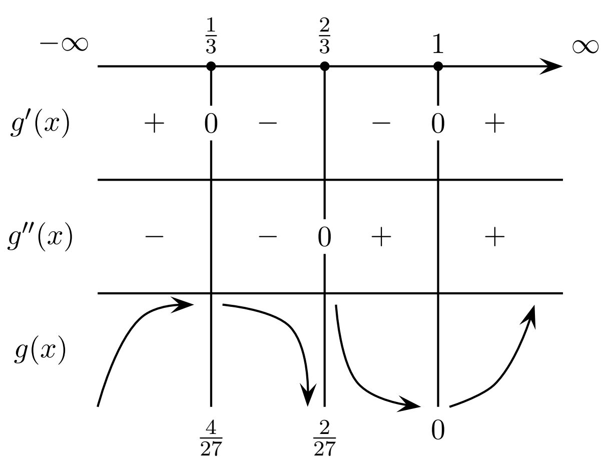
Next we calculate the critical points of g'. Since g' is differentiable everywhere the critical points of g' are the zeros of g''. We have:

$$g''(x) = 6x - 4$$

So

$$g''(x) = 0 \iff x = \frac{2}{3}$$

Now we construct a table that shows the signs of g' and g'', and the behavior of g:



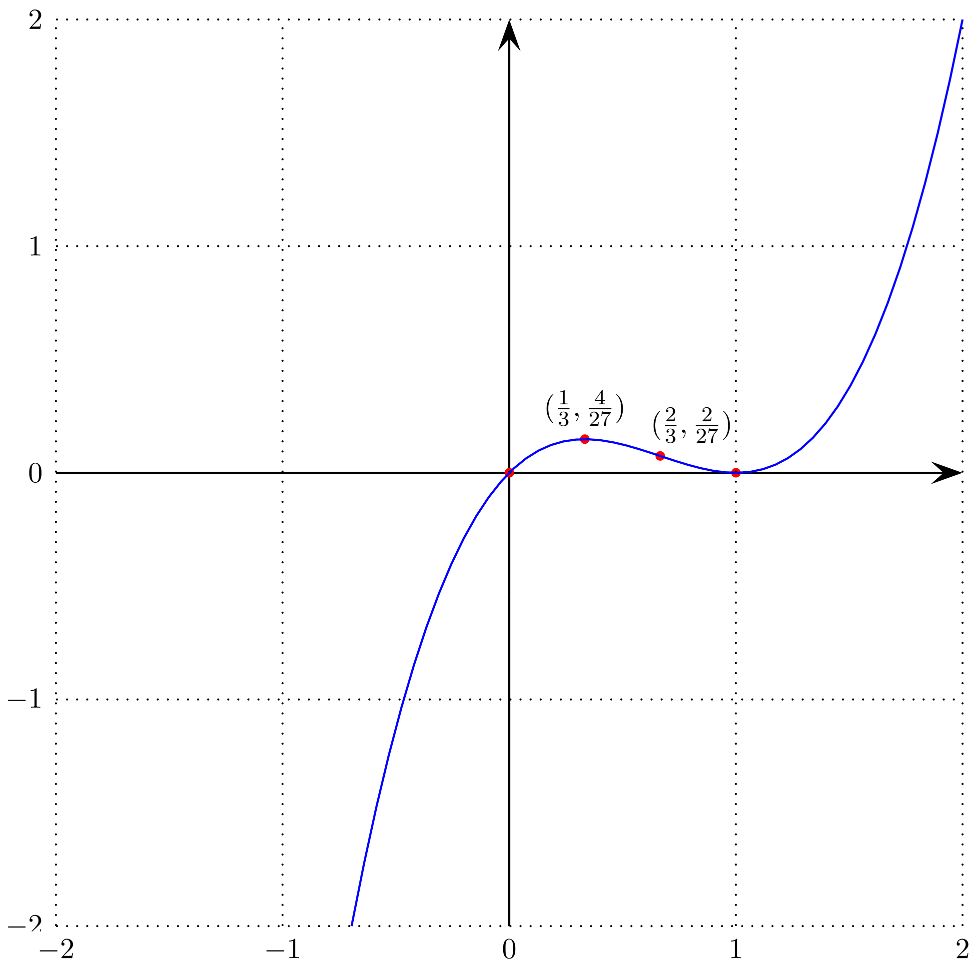


Figure 1: The graph of $y = x^3 - 2x^2 + x$

Putting all this information together we have the following graph in Figure 1 for $g(x)$. From this we get the graph in Figure 2 for $f(x) = |g(x)|$ □

4. Sketch a graph of the function

$$f(x) = \cos x - \sin x$$

The graph should correctly indicate x and y intercepts, local extrema, points of inflection, the intervals where f is increasing or decreasing, and the intervals where f is concave upwards or downwards.

Answer. This is a periodic function with period 2π, so we'll only analyze it in the interval $[0, 2\pi]$. We start by finding the x-intercepts:

$$\begin{aligned} f(x) = 0 &\iff \cos x - \sin x = 0 \\ &\iff \cos x = \sin x \\ &\iff \tan x = 1 \\ &\implies x = \frac{\pi}{4}, \text{ or } x = \frac{5\pi}{4} \end{aligned}$$

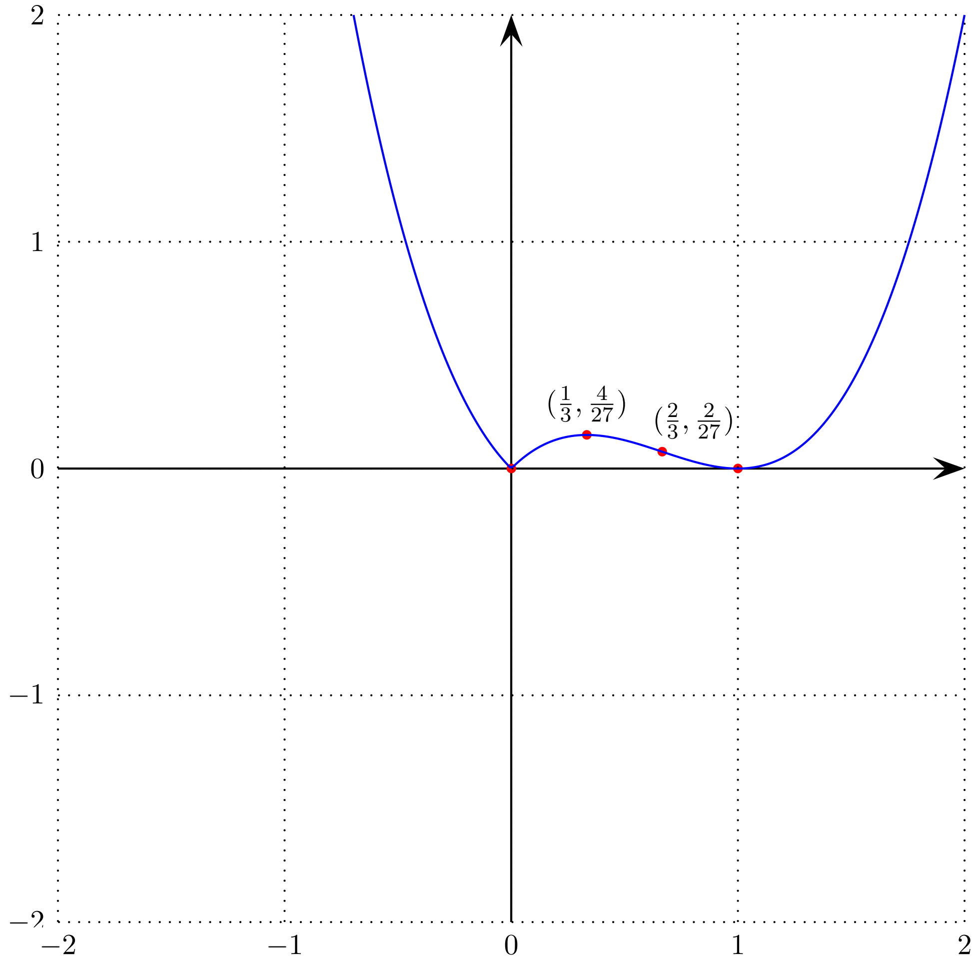


Figure 2: The graph of $y = |x^3 - 2x^2 + x|$

Next we find the critical points of f. We have:

$$f'(x) = -\sin x - \cos x$$

So we have:

$$\begin{aligned} f'(x) = 0 &\implies -\sin x - \cos x = 0 \\ &\implies \sin x = -\cos x \\ &\iff \tan x = -1 \\ &\implies x = \frac{3\pi}{4}, \text{ or } x = \frac{7\pi}{4} \end{aligned}$$

Next we find the critical points of f', that is the zeros of f'':

$$f''(x) = -\cos x + \sin x$$

So we have:

$$\begin{aligned} f'(x) = 0 &\implies -\cos x + \sin x = 0 \\ &\implies \cos x = \sin x \\ &\iff \tan x = 1 \\ &\implies x = \frac{\pi}{4}, \text{ or } x = \frac{5\pi}{4} \end{aligned}$$

We next make a table of values for f:

x	$f(x)$
0	$\cos 0 - \sin 0 = 1$
$\frac{\pi}{4}$	$\cos \frac{\pi}{4} - \sin \frac{\pi}{4} = 0$
$\frac{3\pi}{4}$	$\cos \frac{3\pi}{4} - \sin \frac{3\pi}{4} = -\sqrt{2}$
$\frac{5\pi}{4}$	$\cos \frac{5\pi}{4} - \sin \frac{5\pi}{4} = 0$
$\frac{7\pi}{4}$	$\cos \frac{7\pi}{4} - \sin \frac{\pi}{4} = \sqrt{2}$
2π	$\cos 2\pi - \sin 2\pi = 1$

Now we construct the table of signs:

	0		$\frac{\pi}{4}$		$\frac{3\pi}{4}$		$\frac{5\pi}{4}$		$\frac{7\pi}{4}$		2π
$f'(x)$		$-$		$-$	0	$+$		$+$	0	$-$	
$f''(x)$		$-$	0	$+$		$+$	0	$-$		$+$	
$f(x)$	1	↘	0	↘	$-\sqrt{2}$	↗	0	↗	$\sqrt{2}$	↘	1

where to find the sign of f' we used as test-points the values 0, $\frac{\pi}{4}$, $\frac{5\pi}{4}$, and 2π. To get the sign of f'' we used the test-points 0, $\frac{3\pi}{4}$, $\frac{7\pi}{4}$, and 2π.

So for $0 \le x \le 2\pi$ we have the graph of Figure 3

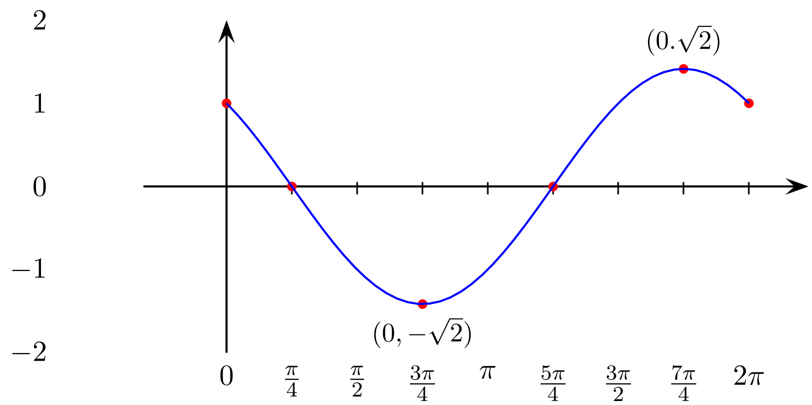


Figure 3: The graph of $y = \cos x - \sin x$ on the interval $[0, 2\pi]$

□

Since f is periodic with period 2π the graph will repeat at intervals of length 2π.

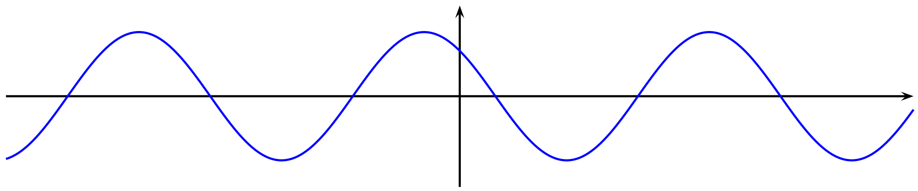


Figure 4: The graph of $y = \cos x - \sin x$

5. Sketch a graph of the function

$$f(x) = \frac{x^2 - 4}{x^2 - 1}$$

The graph should correctly indicate x and y intercepts, local extrema, points of inflection, the intervals where f is increasing or decreasing, the intervals where f is concave upwards or downwards, and any horizontal or vertical asymptotes.

Answer. The domain of f is $\{x : x \neq 1 \text{ and } x \neq -1\}$. We notice that f is an even function, so we'll concentrate on the graph of $y = f(x)$ for $x \geq 0$.

The y-intercept of $y = f(x)$ is at $f(0) = 4$.

We then find the x-intercepts:

$$\begin{aligned} f(x) = 0 &\iff \frac{x^2 - 4}{x^2 - 1} = 0 \\ &\iff x^2 - 4 = 0 \\ &\iff x = -2 \text{ or } x = 2 \end{aligned}$$

Next we find the end behavior; by symmetry we only look at $x \to \infty$:

$$\begin{aligned} \lim_{x \to \infty} f(x) &= \lim_{x \to \infty} \frac{x^2 - 4}{x^2 - 1} \\ &= \lim_{x \to \infty} \frac{1 - \frac{4}{x^2}}{1 - \frac{1}{x^2}} \\ &= \frac{1 - 0}{1 - 0} \\ &= 1 \end{aligned}$$

Thus the line $y = 1$ is a horizontal asymptote.

Next we look at the behavior near the points where f is not defined. Again by symmetry we only look at the behavior near $x = 1$. We have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x^2 - 4}{x^2 - 1} = \frac{1 - 4}{0^-} = \infty$$

and

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x^2 - 4}{x^2 - 1} = \frac{1 - 4}{0^+} = -\infty$$

So the line $x = 1$ is a vertical asymptote.

Next we find the intervals where f is increasing or decreasing and the intervals where the graph is concave upwards or downwards. We have:

$$
\begin{aligned}
f'(x) &= \frac{2x(x^2-1)-(x^2-4)2x}{(x^2-1)^2} \\
&= \frac{2x^3-2x-2x^3+8x}{(x^2-1)^2} \\
&= \frac{6x}{(x^2-1)^2}
\end{aligned}
$$

We notice that the derivative exists for all x in the domain of f, therefore the critical points of f are the solutions to:

$$
\begin{aligned}
f'(x)=0 &\iff \frac{6x}{(x^2-1)^2}=0 \\
&\iff 6x=0 \\
&\iff x=0
\end{aligned}
$$

Next we look at the second derivative:

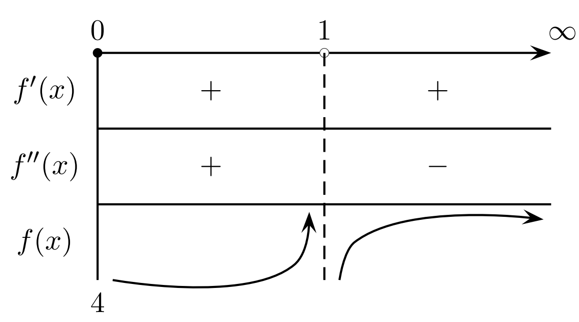
$$
\begin{aligned}
f''(x) &= \frac{6(x^2-1)^2-6x\cdot 2(x^2-1)2x}{(x^2-1)^4} \\
&= \frac{6(x^2-1)-6x\cdot 2\cdot 2x}{(x^2-1)^3} \\
&= \frac{-18x^2-6}{(x^2-1)^3} \\
&= -\frac{6(3x^2+1)}{(x^2-1)^3}
\end{aligned}
$$

We notice that the second derivative exists for all x in the domain of f so that the critical points of f' are the solutions to

$$
\begin{aligned}
f''(x)=0 &\iff -\frac{6(3x^2+1)}{(x^2-1)^3}=0 \\
&\iff 3x^2+1=0
\end{aligned}
$$

Since the last equation has no real solutions the first derivative has no critical points.

Now we find the sign of f' and f''. We have the following table:



So we have the graph of Figure 5 for $y=f(x)$ on $[0,\infty)$: Since f is even its graph is symmetric with respect to the y-axis. So we have the graph of Figure 6 for $y=f(x)$: □

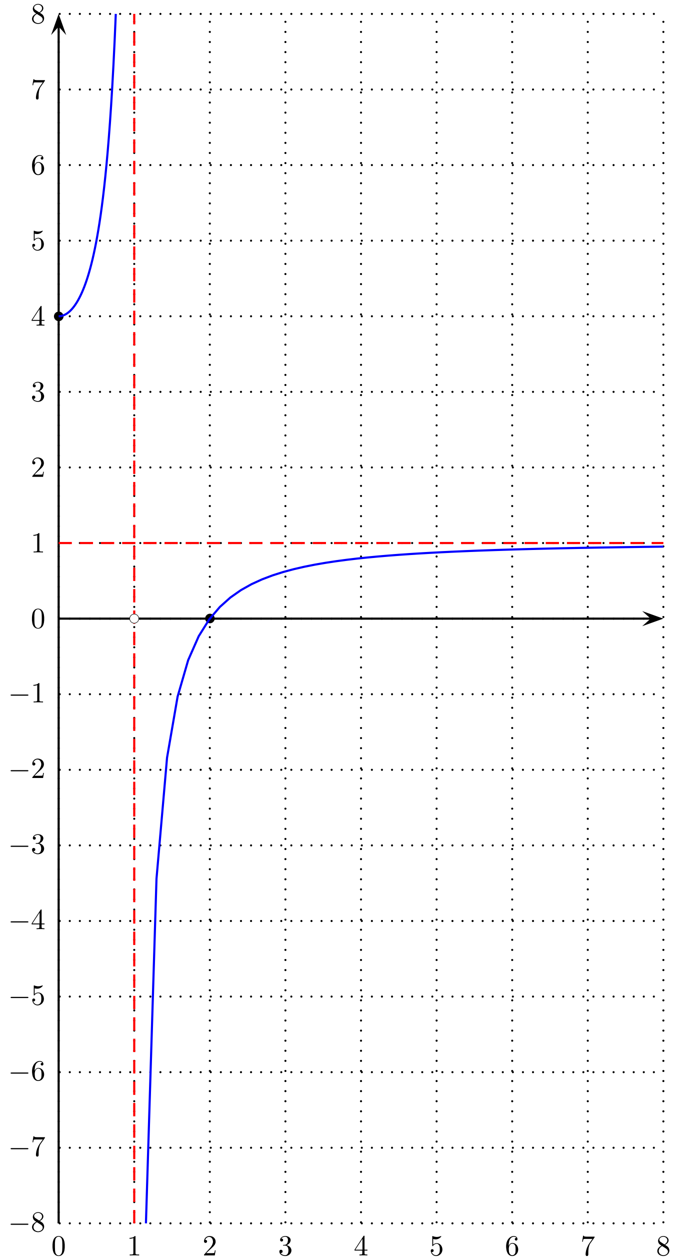


Figure 5: The graph of $y = \dfrac{x^2 - 4}{x^2 - 1}$ for $x \geq 0$

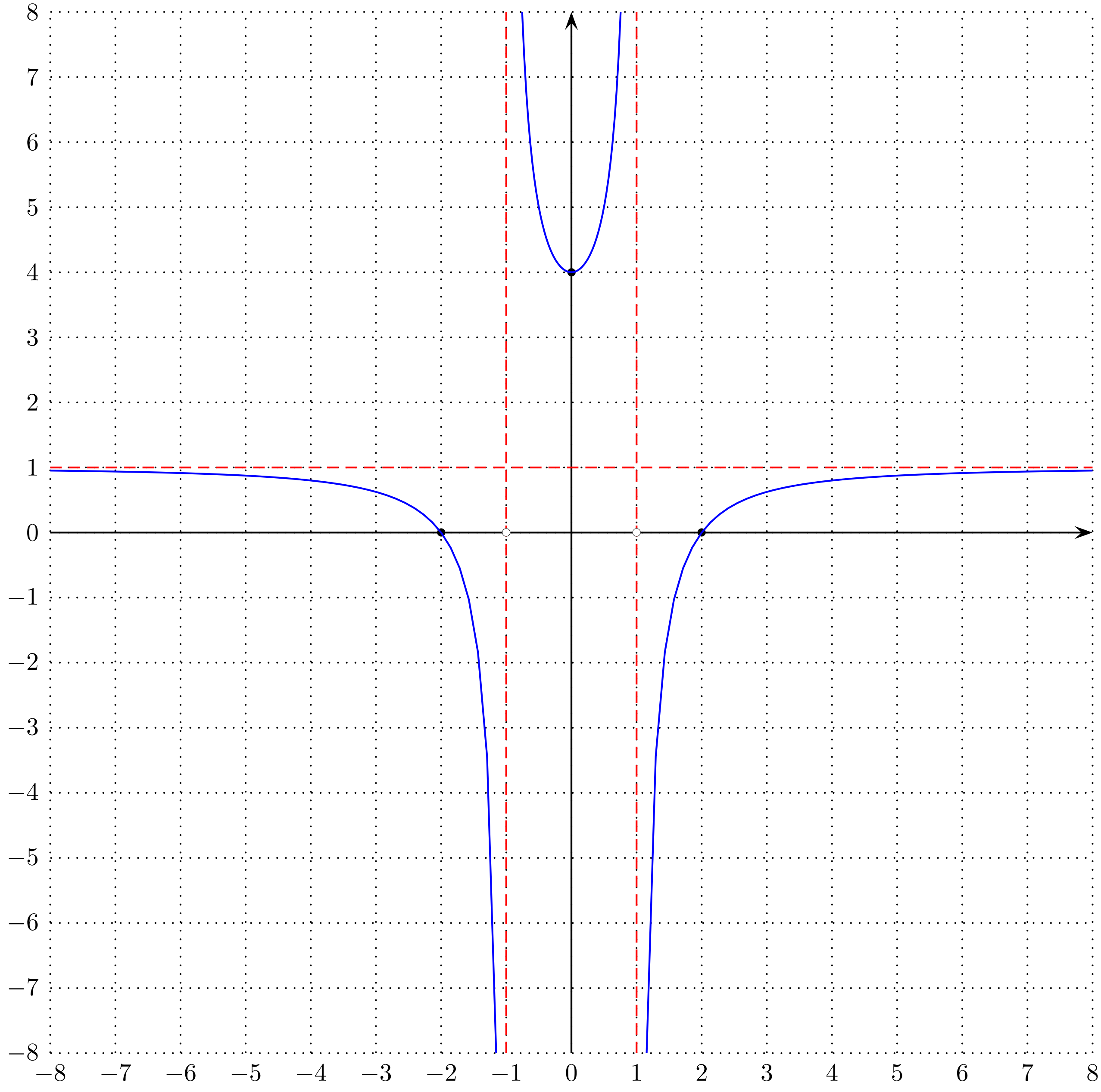


Figure 6: The graph of $y = \dfrac{x^2 - 4}{x^2 - 1}$