BRONX COMMUNITY COLLEGE
of the City University of New York

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

MATH 31 | Exam 1
Nikos Apostolakis | October 20, 2011

The Answers

1. Find the following limits. Your answer should be a real number, $+\infty$, $-\infty$, or *Does Not Exist*.

(a) $\displaystyle\lim_{x\to -5} \frac{x^2-2x-35}{x+5}$

Answer. If we substitute $x=-5$ to $\dfrac{x^2-2x-35}{x+5}$ we get the indeterminate form $\dfrac{0}{0}$. So we'll factor numerator and denominator to "cancel the common zero":

$$\begin{aligned}\frac{x^2-2x-35}{x+5} &= \frac{(x-7)(x+5)}{x+5}\\ &= x-7\end{aligned}$$

So we have:

$$\lim_{x\to -5} \frac{x^2-2x-35}{x+5} = \lim_{x\to -5}(x-7) = -12$$

□

(b) $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{3x-\tan 5x}$

Answer. Again substituting $x=0$ in $\dfrac{\sin 5x}{3x-\tan 5x}$ gives the indeterminate form $\frac{0}{0}$. In this case it is easier to work with the inverse fraction $\dfrac{3x-\tan x}{\sin 5x}$. We have:

$$\begin{aligned}\frac{3x-\tan x}{\sin 5x} &= \frac{3x}{\sin 5x} - \frac{\tan 5x}{\sin 5x}\\ &= \frac{3x}{\sin 5x} - \frac{\frac{\sin 5x}{\cos 5x}}{\sin 5x}\\ &= \frac{3x}{\sin 5x} - \frac{1}{\cos 5x}\end{aligned}$$

Now

$$\lim_{x\to 0}\frac{3x}{\sin 5x} = \lim_{x\to 0}\frac{3}{5}\cdot\frac{5x}{\sin 5x} = \frac{3}{5}\lim_{x\to 0}\frac{5x}{\sin 5x} = \frac{3}{5}\cdot 1 = \frac{3}{5}$$

and

$$\lim_{x\to 0}\frac{1}{\cos 5x} = \frac{1}{\cos 5\cdot 0} = \frac{1}{1} = 1$$

So

$$\lim_{x\to 0}\frac{3x-\tan x}{\sin 5x} = \lim_{x\to 0}\frac{3x}{\sin 5x} - \lim_{x\to 0}\frac{1}{\cos 5x} = \frac{3}{5} - 1 = -\frac{2}{5}$$

Therefore:

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin 5x}{3x-\tan 5x} &= \lim_{x\to 0}\frac{1}{\frac{3x}{\sin 5x}-\frac{1}{\cos 5x}}\\
&= \frac{1}{\lim_{x\to 0}\left(\frac{3x}{\sin 5x}-\frac{1}{\cos 5x}\right)}\\
&= \frac{1}{-\frac{2}{5}}\\
&= -\frac{5}{2}
\end{aligned}$$

□

(c) $\lim_{x\to -7}\frac{|x+7|}{x+7}$

Answer. We have:

$$\frac{|x+7|}{x+7} = \begin{cases} -1 & \text{if } x < -7 \\ 1 & \text{if } x \geq -7 \end{cases}$$

So

$$\lim_{x\to -7^-}\frac{|x+7|}{x+7} = \lim_{x\to -7^-} -1 = -1$$

while

$$\lim_{x\to -7^+}\frac{|x+7|}{x+7} = \lim_{x\to -7^+} 1 = 1$$

Therefore since the two side limits are not equal we have that $\lim_{x\to -7}\frac{|x+7|}{x+7}$ *Does Not Exist.* □

(d) $\lim_{x\to 0}\frac{x^3-6x^2+8x}{x^5-x^4-12x^3}$

Answer. Again we have the indeterminate form $\frac{0}{0}$. We have:

$$\begin{aligned}
\frac{x^3-6x^2+8x}{x^5-x^4-12x^3} &= \frac{x(x^2-6x+8)}{x^3(x^2-x-12)}\\
&= \frac{x(x-2)(x-4)}{x^3(x+3)(x-4)}\\
&= \frac{x-2}{x^2(x+3)}\\
&= \frac{1}{x^2}\frac{x-2}{x+3}
\end{aligned}$$

So

$$\begin{aligned}
\lim_{x\to 0}\frac{x^3-6x^2+8x}{x^5-x^4-12x^3} &= \lim_{x\to 0}\frac{1}{x^2}\frac{x-2}{x+3}\\
&= \lim_{x\to 0}\frac{1}{x^2}\cdot\lim_{x\to 0}\frac{x-2}{x+3}\\
&= \frac{1}{0^+}\cdot\frac{0-2}{0+3}\\
&= \infty\cdot\frac{0-2}{0+3}\\
&= -\infty
\end{aligned}$$

□

2. Prove that the equation $5x^3 - 7x^2 + 8x - 1 = 0$ has a solution in the interval $(0, 1)$.

   Answer. Let $f(x) = 5x^3 - 7x^2 + 8x - 1$. We have to prove that there is a c in $(0, 1)$ so that $f(c) = 0$. Now, f is continuous on the closed interval $[0, 1]$ and $f(0) = -1$ while $f(1) = 5$. Since 0 is between -1 and 5, according to the Intermediate Value Theorem, there is a c in $(0, 1)$ with $f(c) = 0$. □

3. Calculate $\frac{d}{dx}\left(\sqrt{2x+3}\right)$ using the definition of the derivative as a limit of the difference quotients.

   Answer. We have:

$$\begin{aligned}
\frac{d}{dx}\left(\sqrt{2x+3}\right) &= \lim_{h\to 0} \frac{\sqrt{2(x+h)+3} - \sqrt{2x+3}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2(x+h)+3} - \sqrt{2x+3}}{h} \cdot \frac{\sqrt{2(x+h)+3} + \sqrt{2x+3}}{\sqrt{2(x+h)+3} + \sqrt{2x+3}} \\
&= \lim_{h\to 0} \frac{2(x+h)+3-(2x+3)}{h\left(\sqrt{2(x+h)+3} + \sqrt{2x+3}\right)} \\
&= \lim_{h\to 0} \frac{2x+2h+3-2x-3}{h\left(\sqrt{2(x+h)+3} + \sqrt{2x+3}\right)} \\
&= \lim_{h\to 0} \frac{2h}{h\left(\sqrt{2(x+h)+3} + \sqrt{2x+3}\right)} \\
&= \lim_{h\to 0} \frac{2}{\sqrt{2(x+h)+3} + \sqrt{2x+3}} \\
&= \frac{2}{\sqrt{2x+3} + \sqrt{2x+3}} \\
&= \frac{2}{2\sqrt{2x+3}} \\
&= \frac{1}{\sqrt{2x+3}}
\end{aligned}$$

□

4. Calculate the following derivatives. Simplify your answer as much as possible:

   (a) $\left(\frac{(x-3)^2}{x^2-9}\right)'$

   Answer. In this case it is more convenient to first simplify and then differentiate:

$$\frac{(x-3)^2}{x^2-9} = \frac{(x-3)^2}{(x-3)(x+3)} = \frac{x-3}{x+3}$$

So:

$$\begin{aligned}\left(\frac{(x-3)^2}{x^2-9}\right)' &= \left(\frac{x-3}{x+3}\right)' \\ &= \frac{(x-3)'(x+3)-(x-3)(x+3)'}{(x+3)^2} \\ &= \frac{1\cdot(x+3)-(x-3)\cdot 1}{(x+3)^2} \\ &= \frac{x+3-(x-3)}{(x+3)^2} \\ &= \frac{6}{(x+3)^2}\end{aligned}$$

□

(b) $\left(\sqrt{x^2+1}\sin\sqrt{x^2+1}\right)'$

Answer. We have:

$$\begin{aligned}\left(\sqrt{x^2+1}\sin\sqrt{x^2+1}\right)' &= \left(\sqrt{x^2+1}\right)'\sin\sqrt{x^2+1}+\sqrt{x^2+1}\left(\sin\sqrt{x^2+1}\right)' \\ &= \frac{(x^2+1)'}{2\sqrt{x^2+1}}\sin\sqrt{x^2+1}+\sqrt{x^2+1}\cos\sqrt{x^2+1}\left(\sqrt{x^2+1}\right)' \\ &= \frac{2x}{2\sqrt{x^2+1}}\sin\sqrt{x^2+1}+\sqrt{x^2+1}\cos\sqrt{x^2+1}\,\frac{2x}{2\sqrt{x^2+1}} \\ &= \frac{x\sin\sqrt{x^2+1}}{\sqrt{x^2+1}}+x\cos\sqrt{x^2+1}\end{aligned}$$

□

5. Find the equation of the line tangent to the curve

$$y^3+x^3=2xy^2+x-1$$

at the point $(-2,-1)$

Answer. The slope of the tangent line is

$$\left(\frac{dy}{dx}\right)_{x=-2,y=-1}$$

Using implicit differentiation, we differentiate both sides of the equation of the curve:

$$\begin{aligned}y^3+x^3=2xy^2+x-1 &\Longrightarrow \frac{d}{dx}(y^3+x^3)=\frac{d}{dx}(2xy^2+x-1) \\ &\Longrightarrow 3y^2\frac{dy}{dx}+3x^2=2y^2+4xy\frac{dy}{dx}+1 \\ &\Longrightarrow 3y^2\frac{dy}{dx}-4xy\frac{dy}{dx}=2y^2-3x^2+1 \\ &\Longrightarrow (3y^2-4xy)\frac{dy}{dx}=2y^2-3x^2+1 \\ &\Longrightarrow \frac{dy}{dx}=\frac{2y^2-3x^2+1}{3y^2-4xy}\end{aligned}$$

So:

$$\left(\frac{dy}{dx}\right)_{x=-2,y=-1} = \frac{2(-1)^2 - 3(-2)^2 + 1}{3(-1)^2 - 4(-2)(-1)} = \frac{-9}{-5} = \frac{9}{5}$$

So the equation of the line tangent to the given curve at the point $(-2, -1)$ is

$$y + 1 = \frac{9}{5}(x + 2)$$

or equivalently

$$y = \frac{9x}{5} + \frac{18}{5} - 1$$

which finally simplifies to:

$$y = \frac{9x}{5} + \frac{13}{5}$$

□

6. A particle moves on a vertical line according to the law of motion

$$s(t) = t^3 - 6t^2 + 9t + 5, \qquad t \geq 0$$

where t is measured in seconds and s in meters.

(a) When is the particle moving upward and when is it moving downward?

(b) When is the particle speeding up and when is it slowing down?

(c) Find the total distance traveled by the particle in the first four seconds.

Answer. We first calculate the velocity and the acceleration of the particle. We have that the velocity $v(t)$ is

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$$

and the acceleration is

$$a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt} = 6t - 12$$

We now answer the questions:

(a) The particle is moving upwards when the velocity is positive and downward when the velocity is negative. Now since

$$v(t) = 3(t^2 - 4t + 3)$$

the velocity has the same sign as $f(t) := t^2 - 4t + 3$. We can find the sign of $f(t)$ in two ways:

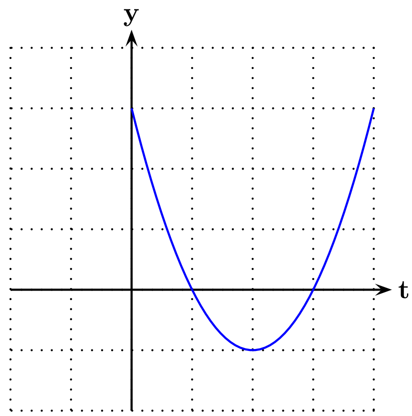
Graphing $y = f(t)$: By completing the square we have

$$\begin{aligned} y = t^2 - 4t + 3 &\iff y - 3 = t^2 - 4t \\ &\iff y - 3 + 4 = t^2 - 4t + 4 \\ &\iff y + 1 = (t - 2)^2 \end{aligned}$$

So the graph of $y = f(t)$ is an upward looking parabola with vertex at $(2, -1)$. We also find the t–intercepts by substituting $y = 0$ in the equation:

$$0 + 1 = (t - 2)^2 \iff \pm 1 = t - 2 \iff t = 3 \text{ or } t = 1$$

So we have the following graph for $y = f(t)$



From the graph we see that $f(t)$, and therefore $v(t)$ as well, is positive for t in $[0, 1)$ or $(3, \infty)$ and negative for t in $(1, 3)$.

Using a table of signs: We first factor $v(t)$:

$$v(t) = 3(t - 1)(t - 3)$$

and then we determine the sign of $v(t)$ by determining the sign of each of each factors using the following table (since 3 is always positive we don't include it in the table):

	0		1		3		∞
$t-1$		$-$	0	$+$		$+$	
$t-3$		$-$		$-$	0	$+$	
$3(t-1)(t-3)$		$+$	0	$-$	0	$+$	

So again, of course, we see that $v(t)$ is positive for t in $[0, 1)$ or $(3, \infty)$ and negative for t in $(1, 3)$.

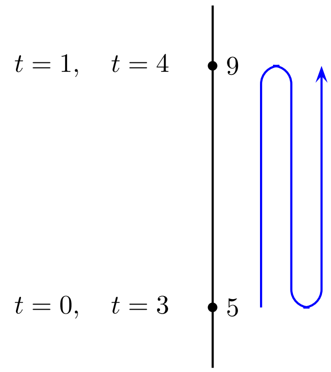
So the particle is moving upwards for $0 \leq t < 1$ and for $t > 3$ and downward for $1 < t < 3$.

(b) The particle is speeding up when the velocity and acceleration have the same sign. We already know the sign of the velocity from the previous step. Now since $a(t) = 6t - 12$ we have that $a(t) < 0$ for $t < 2$ and $a(t) > 0$ for $t > 2$. We now make a combined table of signs for the velocity and acceleration so we can compare their signs:

	0		1		2		3		∞
$v(t)$		$+$	0	$-$		$-$	0	$+$	
$a(t)$		$-$		$-$	0	$+$		$+$	

So the particle is speeding up for $1 < t < 2$ and $t > 3$ and is slowing down for $0 \leq t < 1$ and $2 < t < 3$.

(c) Taking into account part (a), the particle at time $t = 0$ is at $s(0) = 5$, then is moving upwards until $t = 1$ where it is at $s(1) = 9$ then changes direction and moves downward until $t = 3$ where it is at $s(3) = 5$, and then changes direction again and moves upward for $t > 3$, and at $t = 4$ is at $s(4) = 9$. Schematically we have the following diagram for the movement of the particle in the first 4 seconds:



So the total distance traveled by the particle in the first 4 seconds is $3 \cdot (9 - 5) = 12$ meters.

□

7. Use linear approximation to estimate $\sqrt[3]{7.97}$.

   Answer. Let $f(x) = \sqrt[3]{x}$. Then we're asked to approximate $f(7.97)$. Since for $a = 8$ we know $f(a) = \sqrt[3]{8} = 2$ we'll use the linear approximation of f at $a = 8$. The formula for the linearization of f at $a = 8$ is:

   $$L(x) = f(8) + f'(8)(x - 8)$$

   Now

   $$f'(x) = \left(\sqrt[3]{x}\right)' = \left(x^{1/3}\right)' = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

   So

   $$f'(8) = \frac{1}{3\sqrt[3]{64}} = \frac{1}{3 \cdot 4} = \frac{1}{12}$$

   So the linearization of f at $a = 8$ is

   $$L(x) = 2 + \frac{1}{12}(x - 8)$$

   Evaluating the linearization at $x = 7.97$ gives

   $$L(7.97) = 2 + \frac{1}{12}(-0.03) = 1.9975$$

   So we have

   $$\sqrt[3]{7.97} \approx 1.9975$$

□