1 Basic Trigonometric identities

Theorem 1.1. *The following identities hold, whenever all the terms are defined:*

$$\sin^2\theta + \cos^2\theta = 1 \tag{1}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{2}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{3}$$

Remark. We will see soon that identities (2) and (3) follow from (1) and the identities in Theorem 1.2.

Theorem 1.2. *The following identities hold whenever all the terms in them are defined:*

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{4}$$

$$\sec\theta = \frac{1}{\cos\theta} \tag{5}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \tag{6}$$

$$\csc\theta = \frac{1}{\sin\theta} \tag{7}$$

Proof of Theorem 1.1. To prove Identity (1), consider the right triangle in Figure 1 and apply the Pythagorean theorem. Similarly to prove Identity (2) consider the triangle in Figure 2, and to prove Identity (3) consider the triangle in Figure 3.

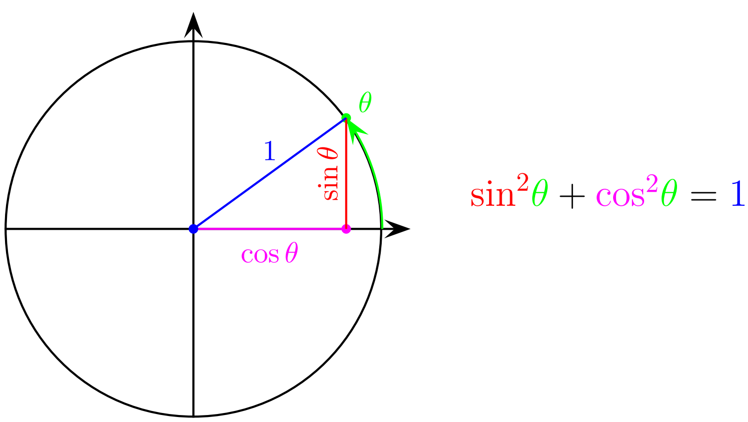
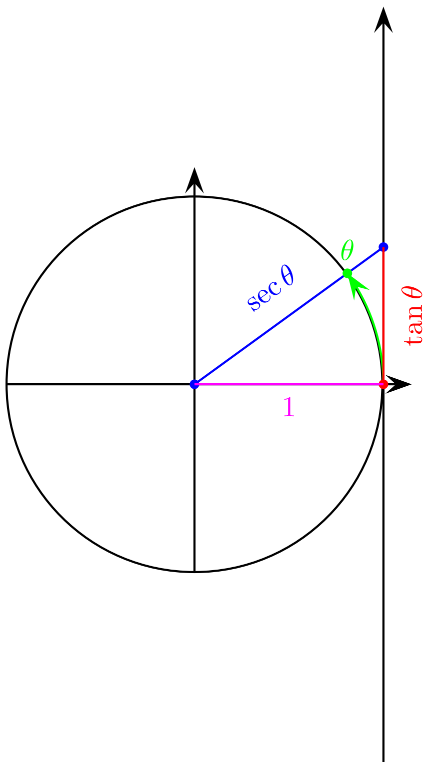


Figure 1: Proof of the first Pythagorean identity

□

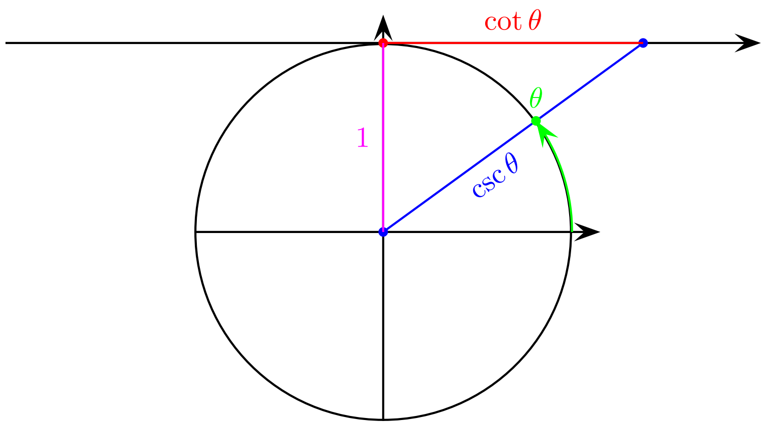
Proof of Theorem 1.2. For an arc θ starting at $A = (1,0)$, let P be the endpoint of θ, and let O be the the origin. Also let T be the point where the radius to P hits the line tangent to the circle at A, and C be the point where the vertical line from P meets the x–axis (see Figure 4). Then triangles OCP and OAT are similar so the ratios of their corresponding sides are equal. So we have:

$$\frac{AT}{OA} = \frac{CP}{OC}$$



$$1 + \tan^2\theta = \sec^2\theta$$

Figure 2: Proof of the second Pythagorean identity



$$1 + \cot^2\theta = \csc^2\theta$$

Figure 3: The proof of the third Pythagorean identity

or equivalently:

$$\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta}$$

So:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

and we have proved (4).

We also have:

$$\frac{OT}{OA} = \frac{OP}{OC}$$

or equivalently

$$\frac{\sec\theta}{1} = \frac{1}{\cos\theta}$$

so:

$$\sec\theta = \frac{1}{\cos\theta}$$

and we have proved (5).

Equations (6) and (7) are proved in an entirely similar manner. $\square$

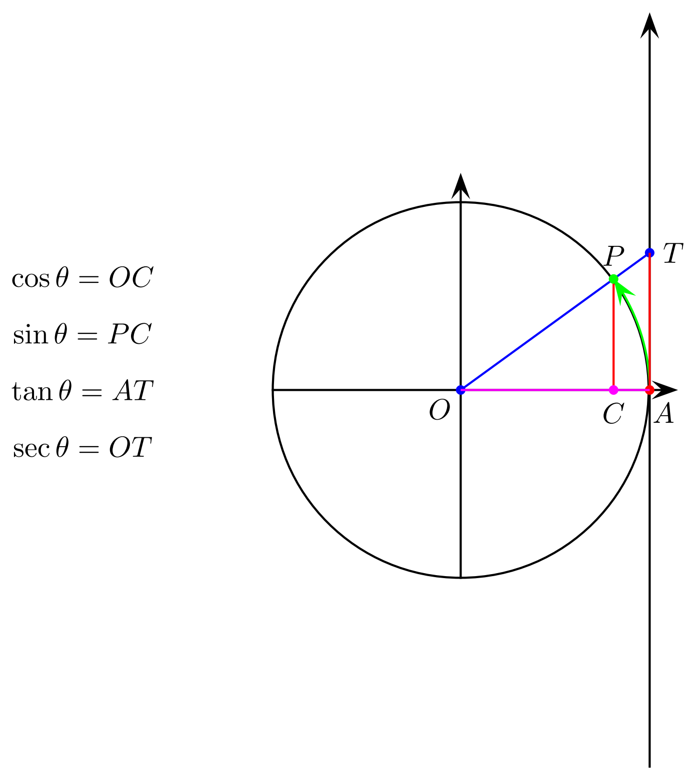


Figure 4: Proof of the second Pythagorean identity

2 Exercises

1. Prove the second and third Pythagorean identities (equations (2 and (3)) using the first Pythagorean Identity (equation (1)) and the identities in Theorem 1.2.

2. Prove the following identities. (Note that we worked some of them in class.)

   (a) $\sec x \cot x = \csc x$

   (b) $\tan^2 x + 1 = \sec^2 x$

   (c) $\tan x + \cot x = \sec x \csc x$

   (d) $\sin x \tan x + \cos x = \sec x$

   (e) $\csc x - \sin x = \cot x \cos x$

   (f) $\dfrac{\cos x}{1 - \sin x} = \sec x + \tan x$

   (g) $\sin x - \sin x \cos^2 x = \sin^3 x$

   (h) $\dfrac{\sin x}{1 + \cos x} = \dfrac{1 - \cos x}{\sin x}$

   (i) $\csc x - \cos x \cot x = \sin x$

   (j) $\dfrac{1}{1 + \sin x} + \dfrac{1}{1 - \sin x} = 2 + 2\tan^2 x$

   (k) $\cos^2 x - \sin^2 x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

   (l) $\cot^2 x - \cos^2 x = \cot^2 x \cos^2 x$

3. Every time we have an expression involving trigonometric functions we can get a *dual* expression by interchanging sine with cosine, tangent with cotangent, and secant with cosecant. It turns out that the dual of trigonometric identity is also a trigonometric identity. Prove the duals of all the trigonometric identities of the previous exercise.