Homework on Polynomial Functions II

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Please note: You should fully justify your answers.

1. What is the remainder of the following division?

$$\frac{4x^{100} - 32x^{41} + x + 7}{x + 1}$$

2. Solve the following polynomial equations.
   (a) $x^3 + 6x^2 - x - 30 = 0$
   (b) $x^3 - 5x - 12 = 0$
   (c) $x^4 + 3x^3 - 16x^2 + 19x - 7 = 0$
   (d) $x^3 + 9x^2 + 27x + 27 = 0$
   (e) $3x^3 - x^2 + 3x - 1 = 0$
   (f) $x^4 + x^3 - 7x^2 - x + 6 = 0$
   (g) $x^4 + x^3 - 11x^2 + 9x - 180 = 0$
   (h) $x^5 - x^4 - 5x^3 + x^2 + 8x + 4 = 0$
   (i) $x^4 - 7x^3 + 13x^2 + 3x - 18 = 0$
   (j) $x^8 - 2x^7 - 9x^6 + 12x^5 + 27x^4 - 18x^3 - 31x^2 + 8x + 12 = 0$
   (k) $6x^3 + 41x^2 - 8x - 7 = 0$
   (l) $10x^4 + 29x^3 - 15x^2 - 5x + 2 = 0$
   (m) $12x^4 + 92x^3 + 43x^2 - 88x + 21 = 0$
   (n) $10x^6 - 19x^5 + 6x^4 - 10x^2 + 19x - 6 = 0$

3. Sketch a rough graph for each of the following polynomial functions. The graph should correctly reflect the end behavior, the behavior near x–intercepts and the number of turning points. The y–intercept should also be correctly marked.
   (a) $p(x) = x^4 + 4x^3 + 6x^2 + 4x + 1$
   (b) $g(x) = x^3 - 6x^2 + 12x - 8$
   (c) $h(x) = 6x^3 - x^2 - 11x + 6$
   (d) $k(x) = x^4 - 11x^2 + 24$
   (e) $f(x) = -2x^4 + 4x^3 + 22x^2 - 24x - 72$
   (f) $f(x) = x^5 + 2x^4 - 6x^3 - 8x^2 + 5x + 6$
   (g) $g(x) = x^4 - 5x^3 + x^2 + 21x - 18$

4. Solve the following inequalities: (you may use the results from the previous exercise).
   (a) $x^5 + 2x^4 - 6x^3 - 8x^2 + 5x + 6 \leq 0$
   (b) $x^3 - 6x^2 + 12x - 8 \geq 0$
   (c) $x^4 - 5x^3 + x^2 + 21x - 18 < 0$
   (d) $-2x^4 + 4x^3 + 22x^2 - 24x - 72 > 0$
   (e) $x^4 + 4x^3 + 6x^2 + 4x + 1 \leq 0$

5. Find a fourth degree polynomial with real coefficients with roots at $x = 1 - 2i$ and $x = 3i$.

6. For each of the following lists of properties, give an example of a polynomial $p(x)$ that has all of the properties.
   (a) The degree of $p(x)$ is 3 and its graph intercepts the x–axis at the points $x = 0$, $x = 1$ and $x = 3$. Additionally as $x \to \infty$, $p(x) \to -\infty$.
   (b) The degree of $p(x)$ is 3. The zeros of $p(x)$ are $-1, 2, 3$ and its constant term is 12.
   (c) The only x-intercepts of $y = p(x)$ are $x = -3$, $x = 1$, and $x = 2$. As $x \to \infty$, $p(x) \to \infty$ and as $x \to -\infty$, $p(x) \to \infty$. The y–intercept of $y = p(x)$ is at $y = 18$.
   (d) The solution set of the inequality $p(x) < 0$ is empty and the polynomial has exactly two real roots $x = 1$ and $x = -1$. Additionally the leading coefficient is 4 and the constant term is 8.

7. Prove that a polynomial with real coefficients and odd degree has at least one real root.

8. For each of the following real numbers a
   1. Find a polynomial with integer coefficients that has a as a root.
   2. Prove that a is irrational.

   (a) $a = \sqrt{11}$
   (b) $a = \sqrt[5]{4}$
   (c) $a = 2 - \sqrt{3}$
   (d) $a = \sqrt{3} - \sqrt{2}$

9. All the roots of the following equation are rational numbers:

$$x^5 + ax^4 + bx^3 + cx - 5$$

   where a, b, c are integers. Prove that this equation has at least one multiple root.