BRONX COMMUNITY COLLEGE
of the City University of New York

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

MATH 30 | Exam 1
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The Answers

1. For the function f whose graph is shown bellow, find:
   (a) The domain of f.

   Answer. $[6, 7)$ $\square$

   (b) The range of f.

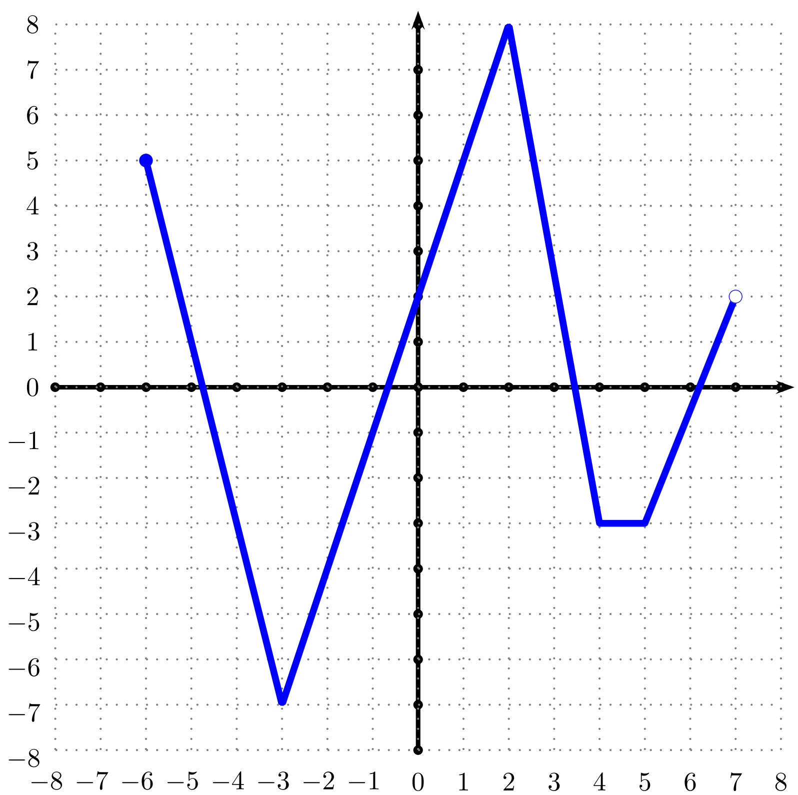
   Answer. $[-7, 8]$ $\square$

   (c) The intervals where f is increasing.

   Answer. $[-3, 2]$ and $[5, 7)$ $\square$

   (d) The intervals where f is decreasing.

   Answer. $[-6, -3]$ and $[2, 4]$. $\square$



2. Find the domain of each of the following functions:
   (a) $f(x) = \sqrt{3 - 6x}$

Answer. Only non-negative numbers have (real) square roots. So in order for f to be defined we need $3 - 6x \geq 0$, or equivalently, $3 \geq 6x$, or, $\frac{1}{2} \geq x$. Thus the domain of f is $\left(-\infty, \frac{1}{2}\right]$. □

(b) $g(x) = \dfrac{2x-1}{x^2 - x - 12}$

Answer. We need the denominator to be non-zero. The denominator factors as

$$x^2 - x - 12 = (x+3)(x-4)$$

so in order for the denominator to be non-zero we need $x \neq -3$ and $x \neq 4$. Thus the domain of g is

$$(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$$

□

3. For the function $f(x) = \dfrac{x+3}{2x-1}$ find

(a) The formula for the inverse function f^{-1}.

Answer. As a relation f is given by:

$$f\colon\ y = \frac{x+3}{2x-1}$$

so the f^{-1} as a relation is given by

$$f^{-1}\colon\ x = \frac{y+3}{2y-1}$$

To find the formula of the inverse function we solve the last equation for y:

$$\begin{aligned} x = \frac{y+3}{2y-1} &\iff x(2y-1) = y+3 \\ &\iff 2xy - x = y + 3 \\ &\iff 2xy - y = x + 3 \\ &\iff (2x-1)y = x+3 \\ &\iff y = \frac{x+3}{2x-1} \end{aligned}$$

Thus we have the formula:

$$f^{-1}(x) = \frac{x+3}{2x-1}$$

□

(b) The domain of f^{-1}.

Answer. From the formula of f^{-1} we see that its domain is

$$\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$$

□

(c) The range of f^{-1}.

Answer. The range of f^{-1} is the same as the domain of f. From the formula of f we see that its domain is

$$\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$$

Therefore the range of f^{-1} is

$$\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$$

□

Remark. In Question 3, f^{-1} turned out to be the same as f, so f is an *involution*. Of course then the domain and range of f^{-1} turned out to be the same.

4. Let $f(x) = \dfrac{2x-3}{5x+1}$ and $g(x) = \dfrac{1}{x}$. Find:

   (a) The formula for $f \circ g$.

   Answer. We have:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= \frac{2\frac{1}{x} - 3}{5\frac{1}{x} + 1} \\ &= \frac{2-3x}{5+x}\end{aligned}$$

□

   (b) The domain of $f \circ g$.

   Answer. In order for x to be in the domain of $f \circ g$ we need tow conditions

   1. x has to be in the domain of g.
   2. $g(x)$ has to be in the domain of f.

   In order for x to be in the domain of g we need $x \neq 0$, so the first condition is satisfied when $x \neq 0$.
   In order for $g(x)$ to be in the domain of f we need $5 + x \neq 0$, or equivalently $x \neq -5$.
   In sum, in order for x to be in the domain of $f \circ g$ we need $x \neq 0$ and $x \neq -5$. Therefore the domain of $f \circ g$ is

$$(-\infty, -5) \cup (-5, 0) \cup (0, \infty)$$

□

5. Verify that $f(x) = \sqrt[3]{x+1}$ and $g(x) = x^3 - 1$ are a pair of inverse functions.

Answer. We need to verify that

1. $f(g(x)) = x$ for all x in the domain of g
2. $g(f(x) = x)$ for all x in the domain of f

The domain of g is $\mathbb{R}$, the set of all real numbers; we have:

$$\begin{aligned}f(g(x)) &= \sqrt[3]{(x^3-1)+1} \\ &= \sqrt[3]{x^3} \\ &= x\end{aligned}$$

The domain of f is $\mathbb{R}$, the set of all real numbers; we have:

$$\begin{aligned} g(f(x)) &= \left(\sqrt[3]{x+1}\right)^3 - 1 \\ &= (x+1) - 1 \\ &= x \end{aligned}$$

Thus, f and g are a pair of inverse functions. ☐

6. Given that $x = -2$ is a solution of the following equation:

$$x^3 + 4x^2 + 5x + 2 = 0$$

solve the equation completely.

Answer. Since $x = -2$ is a zero, $x + 2$ is a factor of the polynomial $x^3 + 4x^2 + 5x + 2$. We perform the division to find the factorization:

$$\begin{array}{l} x^3 + 4x^2 + 5x + 2 = \left(x+2\right)\left(x^2+2x+1\right) \\ \underline{-x^3 - 2x^2} \\ \quad 2x^2 + 5x \\ \quad \underline{-2x^2 - 4x} \\ \qquad x + 2 \\ \qquad \underline{-x - 2} \\ \qquad\quad 0 \end{array}$$

So the given equation is equivalent to

$$(x+2)(x^2+2x+1) = 0$$

which is equivalent to

$$x + 2 = 0 \text{ or } x^2 + 2x + 1 = 0$$

which is equivalent to

$$x = -2 \text{ or } (x+1)^2 = 0$$

which is equivalent to

$$x = -2 \text{ or } x = -1 \text{ or } x = -1$$

So the roots are $x = -2$ and $x = -1$, a double root. ☐

7. Let $f(x) = (x+2)(x+1)^2(1-x)(x-2)^2$.

   (a) Sketch a rough but qualitatively accurate graph of $y = f(x)$. The graph should correctly indicate the x and y intercepts, the end behavior, and the behavior near the roots of the function.

   Answer. The leading term of $f(x)$ is $x \cdot x^2 \cdot (-x) \cdot x^2 = -x^6$. Thus for the end behavior of f we have that as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$.
   f has single roots at $x = -2$ and $x = 1$ and double roots at $x = -1$ and $x = 2$.
   The y–intercept is at $f(0) = 2 \cdot 1^2 \cdot 1 \cdot (-2)^2 = 8$.
   In sum we have the graph in Figure 1. ☐

   (b) Solve the inequality $(x+2)(x+1)^2(1-x)(x-2)^2 < 0$

   Answer. Examining the graph of f we see that $f(x)$ is negative for x in $(-\infty, -2) \cup (1, 2) \cup (2, \infty)$. ☐

8. Find a fourth degree polynomial that has a double root at $x = 1$, and single roots at $x = 2$ and $x = -1$.

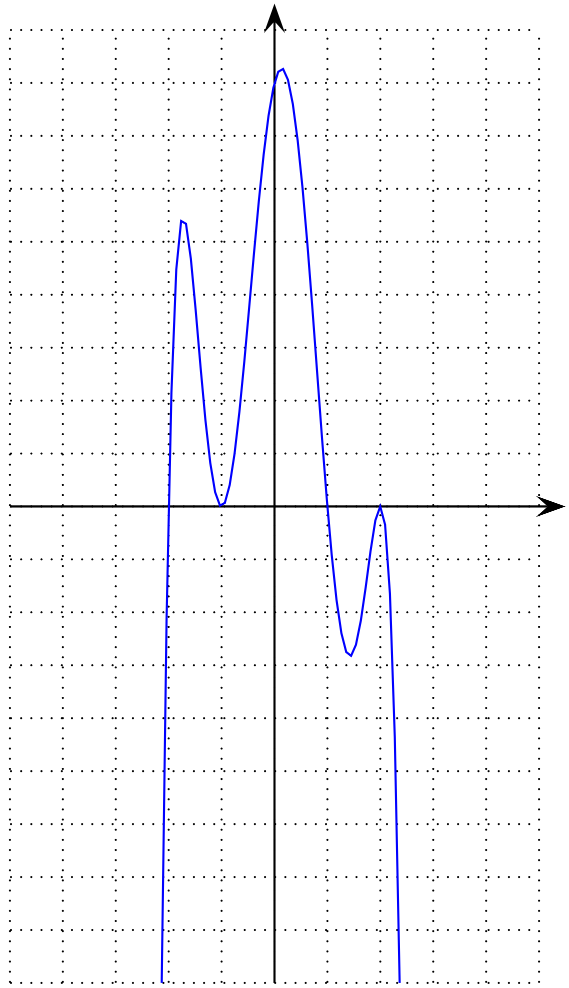


Figure 1: The graph in Question 7(a)

Answer. Such a polynomial has factors $(x-1)^2$, $(x-2)$, and $(x+1)$. So the following polynomial works:

$$\begin{aligned}(x-1)^2(x-2)(x+1) &= (x^2-2x+1)(x^2-x-2) \\ &= x^4-x^3-2x^2-2x^3+2x^2+4x+x^2-x-2 \\ &= x^4-3x^3+x^2+3x-2\end{aligned}$$

□