How to get the Simplified Expanded Form of a polynomial, II

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Recall. Recall that the distributive law states that multiplication distributes over addition and subtraction:

$$a(b \pm c) = ab \pm ac, \qquad (b \pm c)a = ba \pm ca \tag{1}$$

Recall also that going from the LHS to the RHS is referred to as the *expanding* direction of the distributive law, while going from LHS to the RHS is referred to as the *contracting* direction.

Last time we saw how to get the Simplified Expanded Form for sums and differences of polynomials: we use the distributive property in the contracting direction, to combine groups of like terms.

In this section we will see how we can use the expanding direction of the distributive law to get the SEF of the product of two polynomials.

1 The product of two monomials

Recall. Recall the definition of exponents: if a is any real number and n is any natural number then

$$x^n = \underbrace{x \cdot x \cdots x}_{n \text{ factors}}$$

Recall also the following basic properties:

- $x^n x^m = x^{n+m}$
- $(x^m)^n = x^{mn} =$
- $(ax)^n = a^n x^n$

The basic strategy for getting the SEF of the product of two terms is to use the commutative and associative laws for multiplication to write the factors of the product that can be simplified next to each other. In other words, we write the coefficients and all occurrences of each variable next to each other. Let's see some examples:

Example 1. Simplify: $(2x^2)(-3x^4)$.

Solution. We first use the commutative law to bring the coefficients together and the variable together, then we multiply the coefficients and the variable parts:

$$\begin{aligned}(2x^2)(-3x^4) &= (2(-3))\,(x^2x^4)\\ &= (-6)(x^6)\\ &= -6x^6\end{aligned}$$

□

Example 2. Simplify: $(-2x^5y^3)(-3x^4y)$.

Solution. We have:

$$\begin{aligned}(-2x^5y^3)(-3x^4y) &= ((-2)(-3))\,(x^5x^4)(y^3y)\\ &= 6x^9y^4\end{aligned}$$

□

Example 3. Simplify $(5xy^2z^3w^2)(6x^3yw^5)$.

Solution.

$$\begin{aligned}(5xy^2z^3w^2)(6x^3yw^5) &= (5\cdot 6)(xx^3)(y^2y)(z)(w^2w^5)\\ &= 30x^4y^3zw^7\end{aligned}$$

□

Of course, the same method will work if we have the product of more than two terms. For example:

Example 4. Simplify: $3x^2(5xy^2)(-7x^3y)$.

Solution. We have:

$$\begin{aligned}3x^2(5xy^2)(-7x^3y) &= (3\cdot 5(-7))\,(x^2xx^3)(y^2y)\\ &= 105x^6y^3\end{aligned}$$

□

Of course we don't need to explicit perform the second step, in fact, we will usually just perform the calculations in our head and just write down the results.

> To get the simplified form of the product of two or more terms, multiply the numerical coefficients to get the coefficient of the result, and for each variable, add all of the exponents to get the exponent of that variable in the result.

Example 5. Simplify: $2x^3y(-5x^2)(3y^4x)$

Solution. The coefficient of the result is the product of the coefficients, so the the coefficient is $2\cdot(-5)\cdot 3 = -30$. The variable x appears with exponents 3, 2, and 1 so in the result the variable x will appear with exponent $3+2+1=6$, and the variable y appears with exponents 1 and 4 so in the result y will have exponent $1+4=5$. So in sum the simplified form is:

$$-30x^6y^5$$

□

Time to practice:

1. Simplify $7x^5(-3x)$

2. Simplify $(-3xyz)zx$

3. Simplify $ax^3y^2(-by^3x)$

4. Simplify $(-5x^2y^3)(2xz^3)(-2xzy)$

5. Simplify $(-2x^2y^3zw^4)(-3x^3yw^5)(-5z^2w^3x)$

We will often encounter a power of a term. In that case we can use the fact that the power of a product is the product of the powers:

Example 6. Simplify: $(3x^2y)^2$

Solution. We have to take the second power of the monomial $3x^2y$. According to the third property of exponents recalled above, this will equal to the product of the second power of each factor. So:

$$\begin{aligned}(3x^2y)^2 &= 3^2\left(x^2\right)^2y^2\\ &= 9x^4y^2\end{aligned}$$

□

Example 7. Simplify: $(-2x^3yz^2)^3xyz^5$

Solution. In this example we have the product of a power of a monomial with an other monomial. Since exponentiation has priority over multiplication we will first work the power and then multiply:

$$\begin{aligned}(-2x^3yz^2)^3xyz^5 &= (-8x^9y^3z^6)xyz^5\\ &= -8x^{10}y^4z^{11}\end{aligned}$$

□

1. Simplify $(-2x^2y^3)^2xy^2$

2. Simplify $(-3x^2y^2)^2(-xy^3)^3$

3. Simplify $-2xy^4(-5x^2y)^2(2x^4y^7)^3$

2 The product of a monomial and a polynomial

We have already seen how to find the SEF of the product of two monomials. The next step is the product of a monomial and a general polynomial. To do this we use the distributive property in the expanding direction, or as we will briefly say *we expand*. We can state this as a rule:

To expand the product of a term and a polynomial, we multiply the term with each term of the polynomial.

Example 8. Expand $2(3x + 5)$.

Solution. This is the simplest case. We've already seen such examples in the section about linear equations of one variable.

$$2(3x + 5) = 6x + 10$$

□

Example 9. Expand $2x^2y(3x^3 - 5y)$

Solution. We multiply $2x^2y$ first with $3x^3$ to get $6x^5y$, and then with $-5y$ to get $-10x^2y^2$. The expanded form is then the sum of these two terms:

$$2x^2y(3x^3 - 5y) = 6x^5y - 10x^2y^2$$

□

Example 10. Expand $-3x^2(2x^3 - 5x^2 + 2x - 7)$

Solution. Again we multiply $-3x^2$ with each term of $2x^3 - 5x^2 + 2x - 7$ and combine the resulting terms. We get:

$$-3x^2(2x^3 - 5x^2 + 2x - 7) = -6x^5 + 10x^4 - 6x^3 + 21x^2$$

□

Let's practice:

1. Expand $3x(5x - 9)$.

2. Expand $-2x^5(-x^4 + 3x^3 - 2x^2 - 5x + 3)$.

3. Expand $\quad -5x^2y(3x^2y^3 - 5xy^2 - x - 3)$

The following observation may be helpful when we check that we didn't "miss" any term.

Fact 1. *When we expand the product of a term with a polynomial in SEF the result is in SEF and has as many terms as the polynomial. Furthermore the degree of the result is the sum of the degree of the term and the degree of the polynomial.*

1. Without actually carrying out the calculations, find the degree and the number of terms for the expansion of the following:
   (a) $-3x^2(5x^4 - 6x^3 + 5x^2 - 4)$

   (b) $-3x^2y^3(5x^4y - 6x^3y^4 + 5x^2y^7 - 5x^6 + 8y^7)$

2. What kind of polynomial we'll get when we expand the product of a linear term in one variable and a quadratic trinomial in one (the same) variable?

3 The product of two polynomials

To expand the product of two general polynomials we have to use the associative law repeatedly.

Let's start with some examples:

Example 11. Expand $\quad (x + 2)(x + 3)$.

Solution. According to the associative law to multiply something with $(x + 3)$ we have to multiply it first with x and then with 3 and then add the two results. So to multiply $x + 2$ with $x + 3$ we have to first multiply $x + 2$ with x and then multiply $x + 2$ with 3 and add the results. So:

$$(x + 2)(x + 3) = (x + 2)x + (x + 2) \cdot 3$$

Now each of the summands can be further expanded: $(x + 2)x$ expands to $x^2 + 2x$ while $(x + 2)3$ expands to $3x + 6$. Thus the given product expands to

$$x^2 + 2x + 3x + 6$$

This is not in simplified form since there are two like terms (the linear ones). So in order to get the SEF we have to combine the two linear terms to finally get:

$$x^2 + 5x + 6\,.$$

□

Example 12. Expand and Simplify: $(2x - 3y)(2x + 3y)$

Solution. We'll proceed in a similar manner. On the first step we will multiply $2x - 3$ with each term of $2x + 3$, then we will expand each of these products and finally we will combine like terms.

$$\begin{aligned}(2x - 3y)(2x + 3y) &= (2x - 3y)2x + (2x - 3y)3y \\ &= 4x^2 - 6xy + 6xy - 9y^2 \\ &= 4x^2 - 9y^2\end{aligned}$$

□

Example 13. Expand and Simplify: $(x^2 - 5x + 3)(2x - 5)$

Solution. Again, we first multiply $x^2 - 5x + 3$ with each of the terms of $2x - 5$, then we expand the three products we obtain and finally we combine like terms.

$$\begin{aligned}(x^2 - 5x + 3)(2x - 5) &= (x^2 - 5x + 3)2x + (x^2 - 5x + 3)(-5) \\ &= 2x^4 - 10x^2 + 6x - 5x^2 + 25x - 15 \\ &= 2x^4 - 15x^2 + 31x - 15\end{aligned}$$

□

Let's think for a moment what happens when we perform this "repeated expansion" to the product of two polynomials, and for concreteness let's concentrate on the last example. In the first step the first polynomial $x^2 - 5x + 3$ is multiplied with each of the terms of the second polynomial and at the second step we expand each of these (two in our case) products. When we expand the first product ($(x^2 - 5x + 3)2x$) the result is that each term of the first polynomial gets multiplied with the first term of the second polynomial, when we expand the second product ($(x^2 - 5x + 3)(-5)$) the result is that each term or the first polynomial gets multiplied with the second term of the second polynomial. The end result is that *each term of the first polynomial gets multiplied with each term of the second.* Of course, there is nothing special about the two polynomials of the last example, the same thing will happen every time we "repeatedly expand" the product of any two polynomials. So we have the rule:

> To expand the product of two polynomials, we multiply each term of the first with each term of the second.

From now on, we will use this rule to get directly to the result instead of applying repeated expansions. For example:

Example 14. Find the SEF of $(x - 2)(x^2 + 3x)$.

Solution. To expand, we multiply each term of the first with each term of the second. We proceed in a systematic manner, by first multiplying the first term of the first polynomial, namely x with every term of the second polynomial in turn, and then multiplying the second term of the first polynomial, namely -2 with each term of the second polynomial in turn. After expanding we simplify by combining like terms.

$$\begin{aligned}(x-2)(x^2+3x) &= x^3+3x^2-2x^2-6x \\ &= x^3+x^2-6x\end{aligned}$$

□

Example 15. Expand and simplify: $(x-3)(x^2+3x+9)$.

Solution. Again we multiply each term of the first polynomial with each term of the second polynomial in a systematic manner:

$$\begin{aligned}(x-3)(x^2+3x+9) &= x^3+3x^2+9x-3x^2-9x-27 \\ &= x^3-27\end{aligned}$$

□

Example 16. Find the SEF of $(x+2)^2$.

Solution. $(x+2)^2$ means multiply $x+2$ with itself. So we have:

$$\begin{aligned}(x+2)^2 &= (x+2)(x+2) \\ &= x^2+2x+2x+4 \\ &= x^2+4x+4\end{aligned}$$

□

Example 17. Find the SEF of $(x-2)^3$.

Solution. To find $(x+2)^3$ we first find $(x+2)^2$ and then we multiply the result with $x+2$.

$$\begin{aligned}(x-2)^2 &= (x-2)(x-2) \\ &= x^2-2x-2x+4 \\ &= x^2-4x+4\end{aligned}$$

So,

$$\begin{aligned}(x-2)^3 &= (x-2)^2(x-2) \\ &= (x^2-4x+4)(x-2) \\ &= x^3-4x^2+4x-2x^2+8x-8 \\ &= x^3-6x^2+12x-8\end{aligned}$$

□

Time to practice:

1. Expand and Simplify: $(x + 3)(x - 5)$.

2. Expand and Simplify: $(x - 4)(x - 2)$.

3. Expand and Simplify: $(2x + 3)(5x^2 - 2)$.

4. Expand and Simplify: $(x + y)(x - y)$.

5. Expand and Simplify: $(2x + 3)^2$.

6. Expand and Simplify: $(2x - 1)^3$.

7. Expand and Simplify: $(x+2)(x^2-2x+4)$.

8. Expand and Simplify: $(xy+3x^2)(x^3y-5xy+3y^2)$.

9. Expand and Simplify: $(x^2-5)(x^2+5)$

10. Expand and Simplify: $(a+b)^2$.

11. Expand and Simplify: $(x^2-2x+3)(3x^2-5x-2)$.

The following fact may be helpful when we check that we didn't "miss" any term.

Fact 2. *When we expand the product of a polynomial with m terms with a polynomial with n terms the result,* before *simplification, has mn terms. Also the degree of the result equals to the sum of the degrees of the two polynomials.*

1. Consider the following polynomial:

$$(x^3 - 3x^2 + 2x - 4)(2x^4 - 6x^3 + 5x^2 - 4)$$

   After expanding, how many terms will there be before simplification? What is the degree of the result?

2. How many terms will there be in the Simplified Expanded Form of the polynomial of the previous question?

3. How many terms will there be in the expansion of

$$(x - y)^{10}$$

   (a) before simplification?

   (b) after simplification?