How to get the Simplified Expanded Form of a polynomial, I

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Recall. In the last lesson we mentioned that polynomials should always be written in *Simplified Expanded Form*[1], that is, as as sum of *unlike* terms. We also saw two particular ways of ordering the terms of a polynomial in SEF, either in *ascending* or more commonly, in *descending* order. In this, and the following few, lessons we will see how to write any given polynomial in SEF. To do this we will repeatedly use the fact that multiplication distributes over addition: recall the *distributive law*

$$a(b + c) = ab + ac, \qquad (b + c)a = ba + ca$$

This law can be used in two directions. When we go from the LHS to the RHS we will say that we use the *expanding* direction and when we go from the RHS to the LHS we will say that we use the *contracting* direction. Also recall that subtraction is just addition of the opposite and therefore multiplication distributes over subtraction as well:

$$a(b - c) = ab - ac, \qquad (b - c)a = ba - ca$$

Our basic strategy for getting a polynomial in SEF will be:

- First expand, that is, write the polynomial as a sum of terms. (This step will be informally referred to as "getting rid of parentheses").
- Then simplify, that is, combine possible like terms.

We start by explaining the second step first.

0.1 Combining like terms

Let's start with a few simple examples.

Example 1. Find the SEF of the following polynomial:

$$2x^3 + 7x^3$$

Solution. This polynomial is in expanded form since it is the sum of two terms , but the two terms are like so the polynomial is not in *simplified* expanded form. To get it in SEF, we can use the contracting direction of the distributive law, as follows:

$$\begin{aligned} 2x^3 + 7x^3 &= (2 + 7)x^3 \\ &= 9x^3 \end{aligned}$$

□

[1] Or SEF for short.

Example 2. Find the SEF of the following polynomial:

$$4xy^2 - 2xy^2$$

Solution. This polynomial is the difference of two like terms so it is not in simplified expanded form. Using the contracting direction of the distributive law we get:

$$\begin{aligned} 4xy^2 - 2xy^2 &= (4-2)xy^2 \\ &= 2xy^2 \end{aligned}$$

□

Example 3. Find the SEF of the following polynomial:

$$\frac{1}{2}x + 3x$$

Solution. Again we contract:

$$\begin{aligned} \frac{1}{2}x + 3x &= \left(\frac{1}{2} + 3\right)x \\ &= \left(\frac{1}{2} + \frac{6}{2}\right)x \\ &= \frac{7}{2}x \end{aligned}$$

□

Example 4. Find the SEF of the following polynomial:

$$7x^5 - 7x^5$$

Solution. In this case we subtract a term from itself, so the result should be 0. Let's see:

$$\begin{aligned} 7x^5 - 7x^5 &= (7-7)x^5 \\ &= 0x^5 \\ &= 0 \end{aligned}$$

as expected. □

Of course this method is not restricted to only two like terms:

Example 5. Find the SEF of $3x^2 + 5x^2 - 9x^2$.

Solution.

$$\begin{aligned} 3x^2 + 5x^2 - 9x^2 &= (3+5-9)x^2 \\ &= -1x^2 \\ &= -x^2 \end{aligned}$$

□

We don't need to explicitly refer to the distributive law in our calculations. Notice that when we combine two or more like terms, at the end we get a term with the same variable part whose coefficient is obtained by combining the coefficients. For example,

Example 6. Find the SEF of $4x^5 - 7x^5$.

Solution. We have two like terms with variable part x^5. When we combine them we get a term with variable part x^5 and coefficient $4 - 7 = -3$. In other words:

$$4x^5 - 7x^5 = -3x^5$$

□

Let's state this as a rule:

To combine two, or more, like terms, we combine their coefficients and keep the same variable part.

Time to practice:

1. Find the SEF of $11 - 9$

2. Find the SEF of $7x^2 - 5x^2$

3. Find the SEF of $7yx^2 - 6yx^2$

4. Find the SEF of $-3x^3 + 6x^3 - 8x^3$

5. Find the SEF of $8x^2y^3 - 6x^2y^3 - 3x^2y^3$

6. Find the SEF of $-5a^4 + 11a^4 - 6a^4$

Often, in the same polynomial there may be several groups of like terms, with different common variable part for each group. Then to get the SEF we can combine the terms of each group. For example:

Example 7. Find the SEF of $7x^2 - 3x + 5 - 2x - 3 + 2x^2$

Solution. We have three groups of like terms, those with variable part x^2, those with variable part x and those with no variable part, i.e. the constants.

$$7x^2 - 3x + 5 - 2x - 3 + 2x^2$$

The quadratic terms combine to give $9x^2$, the linear terms combine to give $-5x$ and the constant terms combine to give 2. In sum, we have that the SEF of the given polynomial is:

$$9x^2 - 5x + 2$$

□

It is convenient when we actually carry out such calculations to use the *commutative law* of addition to first "collect" all the like terms to be next to each other and then we combine. This reduces the chance of "missing" some terms. It is also advisable to order terms according to their degree. Let's see some examples:

Example 8. Find the SEF of the polynomial $5x^2 - 3x - 9 + 2x^2 + 5x - 2 + 7x$

Solution. We have quadratic, linear, and constant terms. We first collect terms in groups according to degree and then combine the terms in each group.

$$\begin{aligned} 5x^2 - 3x - 9 + 2x^2 + 5x - 2 + 7x &= (5x^2 + 2x^2) + (-3x + 5x + 7x) + (-9 - 2) \\ &= 7x^2 + 9x + (-11) \\ &= 7x^2 + 9x - 11 \end{aligned}$$

□

Example 9. Find the SEF of the polynomial $8x^3 - 7x - 6x^3 + 3 - 4x^2 + 5x^2 - 2 - 2x^3$

Solution. We proceed similarly:

$$\begin{aligned} 8x^3 - 7x - 6x^3 + 3 - 4x^2 + 5x^2 - 2 - 2x^3 &= (8x^3 - 6x^3 - 2x^3) + (-4x^2 + 5x^2) + (-7x) + (3 - 2) \\ &= (0x^3) + (1x^2) + (-7x) + (1) \\ &= x^2 - 7x + 1 \end{aligned}$$

□

Let's practice:

1. Find the SEF of the polynomial: $2x^2 - 3x + 5x^2 + 2x - 1$

2. Find the SEF of the polynomial: $7xy^2 - 8x^2y + 3xy^2 + 2x^2y$

3. Find the SEF of the polynomial: $2ab - 3ba^2 + 4ba - 6a^2b + 3ab$

4. Find the SEF of the polynomial: $-3x^2 + 5x - 7 + 4x + 5x^2 + 3 - 9x + 4 - 2x^2$

0.2 Finding the SEF of sums, opposites, and differences

In this section we will see how to find the Simplified Expanded form of the sums and differences of polynomials.

0.2.1 The sum of two polynomials

In the previous section we have seen how to find the SEF of the sum of two terms: If the terms are like we just combine them, if they are not like then the sum is already in Simplified Expanded Form.

For more general polynomials the procedure is similar: to find the SEF of the sum of two polynomials we just combine all like terms. Let's see some examples:

Example 10. Find the SEF of the sum of $x^3 - 5x + 1$ and $2x^4 - 3x^3 + 4$.

Solution. We first collect the like terms from both polynomials and then we combine them:

$$\begin{aligned}(x^3 - 5x + 1) + (2x^4 - 3x^3 + 4) &= 2x^4 + (x^3 - 3x^3) - 5x + (4 + 1) \\ &= 2x^4 + (-2x^3) - 5x + (5) \\ &= 2x^4 - 2x^3 - 5x + 5\end{aligned}$$

□

Example 11. Let $p(x) = 5x^2 - 3x + 5$ and $q(x) = -4x^2 + 5x - 7$. Find the SEF of $p(x) + q(x)$.

Solution.

$$\begin{aligned}p(x) + q(x) &= (5x^2 - 3x + 5) + (-4x^2 + 5x - 7) \\ &= (5x^2 - 4x^2) + (-3x + 5x) + (5 - 7) \\ &= (x^2) + (2x) + (-2) \\ &= x^2 + 2x - 2\end{aligned}$$

□

Example 12. Expand and simplify: $(10x^2y - 5x^3 + y^2 - x^2) + (7x^2y - 3x^2 - y^2)$

Solution.

$$\begin{aligned}(10x^2y - 5x^3 + y^2 - x^2) + (7x^2y - 3x^2 - y^2) &= (10x^2y + 7x^2y) - 5x^3 + (-x^2 - 3x^2) + (y^2 - y^2) \\ &= 17x^2y - 5x^3 - 4x^2\end{aligned}$$

□

Let's practice:

1. Find the SEF of $p(x) + q(x)$ if $p(x) = 3x^2y^3 - 5x^2y^2 - 5xy + 3$ and $q(x) = 7x^2y^3 + 6xy + 2$

2. Write the sum of $3x^2 - 3x - 7$ and $-7x^2 + 5x + 3$ in SEF.

3. Expand and Simplify: $(-3x^4 + 5x^3 + 4x^2 - 2x + 6) + (4x^4 - 6x^3 - 3x^2 + x - 4)$

0.2.2 The opposite of a polynomial

As we have seen, subtraction is really addition of the opposite. So before we go on to subtraction we should discuss that notion. The *opposite* of a polynomial is an new polynomial, that when we add it to the original polynomial the result is 0. For example:

Example 13. The opposite of $5x^2$ is $-5x^2$ because $5x^2 + (-5x^2) = (5-5)x^2 = 0$. The opposite of $-7x^5$ is $7x^5$ because $-7x^2 + 7x^2 = 0$. Similarly, the opposite of -2 is 2, the opposite of $11x^2y^3$ is $-11x^2y^3$ and the opposite of $-x$ is x.

To find the opposite of a polynomial in SEF is also rather straightforward. For example:

Example 14. The opposite of $6x^2 - 3x$ is $-6x^2 + 3x$ because:

$$\begin{aligned} 6x^2 - 3x + (-6x^2 + 3x) &= (6x^2 - 6x^2) + (-3x + 3x) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Similarly the opposite of $-5xy^2 + 3x^2 - 4xy$ is $5xy^2 - 3x^2 + 4xy$, because when we add these two polynomials we get three pairs of like terms and each such pair gives a 0:

$$\begin{aligned} -5xy^2 + 3x^2 - 4xy + (5xy^2 - 3x^2 + 4xy) &= (-5xy^2 + 5xy^2) + (3x^2 - 3x^2) + (-4xy + 4xy) \\ &= 0 + 0 + 0 \\ &= 0 \end{aligned}$$

In general we have the following fact:

Fact 1. *The opposite of a term, has the same variable part and opposite coefficient. More generally, the opposite of a polynomial that is written as a sum of terms is the sum of the opposite terms.*

Justification. When we add two terms with the same variable part, we just add the coefficients while the variable part stays the same. So if we add two terms with the same variable part and opposite coefficients we get 0. □

As with numbers we use the negative sign in front of a polynomial to denote the opposite of that polynomial; so instead of writing "the opposite of $3x-5$" we write $-(3x-5)$, and instead of writing "the opposite of $-x^2+3x-6$ is x^2-3x+6" we write "$-(-x^2+3x-6)=x^2-3x+6$".

Example 15. Find the SEF of $-(23x^4-5x^3-34x^2+4x-3)$.

Solution. A negative sign in front of a polynomial means the opposite polynomial. So we are asked to find the SEF of the opposite of $23x^4-5x^3-34x^2+4x-3$. According to Fact 1, we only need to change the signs of all the terms. Thus:

$$-(23x^4-5x^3-34x^2+4x-3)=-23x^4+5x^3+34x^2-4x+3$$

□

Let's practice:

1. Put each of the following polynomials in SEF:
   (a) $-(2x^3y-3x^2y^2-7xy^3+2xy)$

   (b) $-(-2x^3+5x^2-23)$

0.2.3 The difference of two polynomials

Subtraction is addition of the opposite. So by putting together the previous two subsections we already know how to find the SEF of the difference of two polynomials. Let's see some examples:

Example 16. Subtract $7x^2-3x-5$ from $2x^3-5x^2+7x+2$ and give your result in Simplified Expanded Form.

Solution. To subtract $7x^2-3x-5$ means to add its opposite, i.e., to add $-7x^2+3x+5$. So we have:

$$\begin{aligned}(2x^3-5x^2+7x+2)-(7x^2-3x-5)&=(2x^3-5x^2+7x+2)+(-7x^2+3x+5)\\&=2x^3-12x^2+10x+7\end{aligned}$$

□

Example 17. Expand and Simplify: $(-2x^2+3x-5)-(-5x^2+7x-3)$.

Solution.

$$\begin{aligned}(-2x^2+3x-5)-(-5x^2+7x-3)&=(-2x^2+3x-5)+(5x^2-7x+3)\\&=3x^2-4x-2\end{aligned}$$

□

Example 18. Let $p(x)=x^2-3x+7$ and $q(x)=-2x^2-3x+8$. Find $p(x)-q(x)$ and $q(x)-p(x)$.

Solution. We have:

$$\begin{aligned}
p(x) - q(x) &= (x^2 - 3x + 7) - (-2x^2 - 3x + 8) \\
&= (x^2 - 3x + 7) + (2x^2 + 3x - 8) \\
&= 3x^2 + 0 - 1 \\
&= 3x^2 - 1
\end{aligned}$$

On the other hand:

$$\begin{aligned}
q(x) - p(x) &= (-2x^2 - 3x + 8) - (x^2 - 3x + 7) \\
&= (-2x^2 - 3x + 8) + (-x^2 + 3x - 7) \\
&= -3x^2 + 0 + 1 \\
&= -3x^2 + 1
\end{aligned}$$

□

Lets practice. As usual we should give our answers in SEF.

1. Subtract x^2 from $-x^2$.

2. Subtract $-2ab$ from $2ab$.

3. Subtract $x + y$ from $x - y$.

4. Subtract $-3x^3 + 4x^2 - 3x - 8$ from $2x^3 - 6x$.

5. For each of the following pairs of polynomials $p(x)$ and $q(x)$ find both $p(x)-q(x)$ and $q(x)-p(x)$.

   (a) $p(x) = 7x^2 - 5x + 3, \quad q(x) = 8x^2 - 3x - 2.$

   (b) $p(x) = 2xy - 3xy^2 + 5x^2y, \quad q(x) = 5xy - 3xy^2 + 5x^2y$

   (c) $p(x) = 3x^3 + 4x^2 - 2x - 7, \quad q(x) = 2x^3 + 3x^2 - 3x - 8$

6. Based on Practice Question 5 above, what is the relation between $p(x) - q(x)$ and $q(x) - p(x)$? Will this be true for any two polynomials? Justify your answer.