First Quiz for CSI35

Nikos Apostolakis

November 6, 2009

Directions: This quiz is due Thursday Novebe 12, at 4:00 PM.

1. Which of the following relations defined on the set of all people are equivalence relations. Justify your answers:
   (a) $(a, b) \in R$ iff a has the same parents as b.
   (b) $(a, b) \in R$ iff a is parent of b.
   (c) $(a, b) \in R$ iff a lives in the same town as b.
   (d) $(a, b) \in R$ iff a lives one floor above b.
   (e) $(a, b) \in R$ iff a is an acquaintance of b.

2. After Alice and Bob finished playing a game they decided to study for the Discrete Mathematics exam that was coming up. After a while they had the following conversation:

   A: I was looking at the definition of an equivalence relation and it seems redundant.

   B: How so?

   A: Well, it says that an equivalence relation is a relation that is reflexive, symmetric and transitive. Now that's redundant, because I can prove that if a relation is symmetric and transitive then it is reflexive as well. So to be economical we should define an equivalence relation to be a relation that is symmetric and transitive. No need to check for reflexivity, really.

   B: Hm, this sound fishy to me. Somebody would've noticed before. Let me see your proof.

   A: It's very simple really: Let R be a symmetric and transitive relation on a set A. To prove that it is reflexive I need to prove that for all $x \in A$ we have $(x, x) \in R$. So let $x \in R$, chose any $y \in A$ such that $(x, y) \in R$. Then since R is symmetric we have $(y, x) \in R$ also. So we have $(x, y) \in R$ and $(y, x) \in R$, so since R is transitive we conclude that $(x, x) \in R$. So, R is reflexive.

   Bob is thoughtful for a while.

B: Hm, your proof seems valid... Still, I find it hard to believe that nobody had noticed this before. Let me think some more.

Bob thinks some more.

B: Ok, your proof must be wrong because I can produce a counter example. Remember the matrix in exercise 4 in the midterm?

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

This matrix is symmetric and transitive, but not reflexive.

A: (after some thought) Yep, you're right. Still though I can't figure out where is the gap in my proof.

B: Me neither ...

(a) Prove that Bob's counter-example really works.

(b) Can you find the fault in Alice's proof?

3. Consider the following relation on $\mathbb{R}$, the set of real numbers

$$(a, b) \in R \Longleftrightarrow |a| = |b|$$

Prove that R is an equivalence relation.

4. Consider the relation R defined on the set of all positive real numbers as follows:

$$(a, b) \in R \quad \text{iff} \quad \frac{a}{b} \in \mathbb{Q}\,,$$

where $\mathbb{Q}$ stands for the set of rational numbers. Prove that R is an equivalence relation.

5. Let Δ_n be the set of all diagonal $n \times n$ matrices with real elements, i.e. a matrix $A = (a_{ij})$ is in Δ_n iff and only if, $\forall i, j \quad i \neq j \Longrightarrow a_{ij} = 0$. Consider the relation $\cong$ defined on the set M_n of all $n \times n$ matrices by

$$A \cong B \Longleftrightarrow A - B \in \Delta_n$$

(a) Prove that $\cong$ is an equivalence relation.

(b) What is the equivalence class of the identity matrix I_n?

6. Consider the relation defined on the set of ordered pairs of natural numbers (i.e. on the set $\mathbb{N} \times \mathbb{N}$) as follows:

$$(m, n) \cong (k, l) \Longleftrightarrow m + l = k + n$$

(a) Prove that $\cong$ is an equivalence relation.

(b) Find the equivalence class of $(5, 6)$.

7. How many equivalence relations are there on the set $\{1, 2, 3, 4, 5, 6\}$?