Second Quiz for CSI35

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Directions: This quiz is due Thursday October 1, at 4:00 PM. Please *staple* all the papers of your answer together.

1. In Nevereverland they have only stamps worth 5 or 7 cents. Prove that a nevereverlander can send any letter that costs 24 cents or more.

2. The Fina Bocci company breeds worms for fishing. After each worm is two weeks old they cut off its tail which becomes a new worm. The tail grows back in a week, so once a worm becomes two weeks old it produces a new worm every week.

   (a) Assuming that no worm ever dies and that the company starts with one newly "born" worm, find a recursive formula for the the number of worms after n weeks.

   (b) Prove that the formula you found in part a) is correct.

3. For an integer n let g_n be the number of ways that n can be written as a sum of ones, twos, or threes, where the order that the summands are written is important. For example, $g(1) = 1$, $g(2) = 2$ since 2 can be written either as 2 or as $1 + 1$, and $g(3) = 4$ because 3 can be written as $1 + 1 + 1$ or as $1 + 2$ or as $2 + 1$ or 3.

   (a) Find a recursive definition of $g(n)$

   (b) Prove that this recursive definition is correct.

   For the next three questions f_n stands for the nth Fibonacci number.

4. Show that for all $n \in \mathbb{N}$

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^n$$

5. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Prove that for all $n \geq 1$, $A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$.

6. Let $\varphi = \dfrac{1+\sqrt{5}}{2}$, $\bar{\varphi} = \dfrac{1-\sqrt{5}}{2}$.

   (a) Prove that $\forall n \in \mathbb{N} \quad \varphi^n = f_{n-1} + f_n\varphi$ and $\bar{\varphi}^n = f_{n-1} + f_n\bar{\varphi}$.

(b) Prove that

$$\forall n \in \mathbb{N} \quad f_n = \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}}$$

For the next four questions recall that if $\Sigma = \{0, 1\}$ then the elements of Σ^*, i.e. the words on the alphabet Σ, are called *bit strings.*

7. The mirror of a bitstring s is a bitstring $m(s)$ that has the same length as s and differs in every bit from s. For example the mirror of 1010001 is 0101110. Give a recursive definition of $m(s)$ for a bitstring s.

8. Consider the following recursive definition of a set of bitstrings MW.

   - $\emptyset \in MW$, where $\emptyset$ stands for the empty bitstring.
   - If $b \in MW$ then $0b1 \in MW$ and $1b0 \in MW$.

   Give a non-recursive description of the bitstrings in MW.

9. The reverse of a string s is the string obtained by "reading s backwards", for example the reverse of the string "sub" is "bus". The reverse of a string s is denoted by s^R. Give a recursive definition of s^R, for bit strings s.

10. A *palindrome* is a string s such that $s^R = s$, in other words a string that reads the same when we read it backwards. For example the string "bob" is a palindrome.

    (a) Give a recursive definition of the set Π of all bitstrings that are palindromes.

    (b) For a natural number n, how many bitstring palindromes of length n are there? Prove your answer.