First Quiz for CSI35

Nikos Apostolakis

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Directions: This quiz is due Tuesday September 15, at 4:00 PM.

1. Prove that for all natural numbers we have:

$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

2. By experimenting with the first few values of n conjecture a formula for the sum of the first $n+1$ even natural numbers. That is, conjecture a formula for

$$\sum_{i=0}^{n} 2i\,.$$

   Then prove your conjecture using mathematical induction.

3. (a) Prove that for all positive integers n, 5 divides $n^5 - n$.

   (b) Is it true that for all positive n, 4 divides $n^4 - n$?

4. Prove that for all $n \geq 1$ we have:

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$

5. Alice and Bob play a game by taking turns removing 1 or 2 stones from a pile that initially has n stones. The person that removes the last stone wins the game. Alice plays always first.

   (a) Prove by induction that if n is a multiple of 3 then Bob has a wining strategy.

   (b) Prove that if n is not a multiple of 3 then Alice has a wining strategy.

6. Chris and Dominique play a slightly different game. Again each player takes turns removing 1 or 2 stones from a pile that initially has n stones but now, the person that removes the last stone loses the game. Chris plays always first. Analyze this game, that is, find the values of n for which Chris has a winning strategy and the values of n for which Dominique has a winning strategy. You should prove your result.

7. Prove that for all $n \in \mathbb{N}$,

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$$

8. Prove that for all $n \in \mathbb{N}$,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

9. Experiment with the first few values of $n \in \mathbb{N}$ to conjecture a formula for the value of

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n$$

Then prove your conjecture using mathematical induction.