Final for CSI35

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Directions: This is the take home part of the final exam. It is due Thursday December 10. Late exams will *not* be accepted.

1. There is a Christmas Carrol called "The twelve days of Christmas":

   On the first day of Christmas,
   my true love sent to me
   A partridge in a pear tree.

   On the second day of Christmas,
   my true love sent to me
   Two turtle doves,
   And a partridge in a pear tree.

   On the third day of Christmas,
   my true love sent to me
   Three French hens,
   Two turtle doves,
   And a partridge in a pear tree.

   ...

   On the twelfth day of Christmas,
   my true love sent to me
   Twelve drummers drumming,
   Eleven pipers piping,
   Ten lords a-leaping,
   Nine ladies dancing,
   Eight maids a-milking,

Seven swans a-swimming,
Six geese a-laying,
Five golden rings,
Four calling birds,
Three French hens,
Two turtle doves,
And a partridge in a pear tree!

If the pattern continues for all natural numbers,

(a) How many new gifts all together will her true love send her in the n–th day of Christmas?

(b) How many gifts will she have accumulated in the n–th day of Christmas?

2. Prove, by induction or otherwise, that the sum of the cubes of any three consecutive natural numbers is divisible by 9.

3. This question is about the Tower of Hanoi puzzle. The puzzle consists of three pegs and six disks, initially stacked in decreasing size on one of the pegs, say A, as shown in Figure 1.

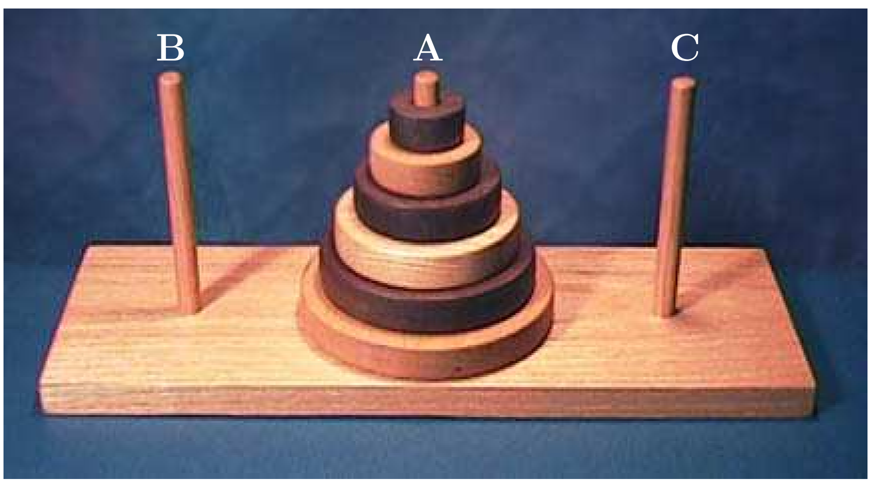

Figure 1: The Towers of Hanoi puzzle

The goal is to transfer the whole tower of six disks to one of the other pegs, say C, moving only one disk at a time and never moving a larger one on top of a smaller.

(a) Prove that this puzzle has a solution for any number of initial disks.

(b) Prove that the puzzle with n disks can be solved in $2^n - 1$ moves.

(c) Prove that the puzzle with n disks cannot be solved in fewer than $2^n - 1$ moves.

4. Prove that for $n \geq 0$:

$$\cfrac{1}{1+\cfrac{1}{\ddots+\cfrac{1}{x}}} = \frac{f_n + f_{n+1}x}{f_{n+1} + f_{n+2}x}$$

where in the Left Hand Side there are $n + 2$ fraction lines and f_n stands for the Fibonacci numbers.

5. Let $\mathbb{N}^*$ be the set of positive integers. The relation $\sim$ on $\mathbb{N}^*$ is defined as follows:

$$m \sim n \iff \exists k \in \mathbb{N}^* \quad mn = k^2$$

   (a) Prove that $\sim$ is an equivalence relation.

   (b) Find the equivalence classes of 2, 4, and 6.

6. For $n = 0, 1, 2, 3, 4$ construct the Hasse diagram of the powerset of a set with n elements. What family of graphs you get? Will this pattern continue? Can you prove it?

7. Prove that the two graphs in Figure 2 are isomorphic.

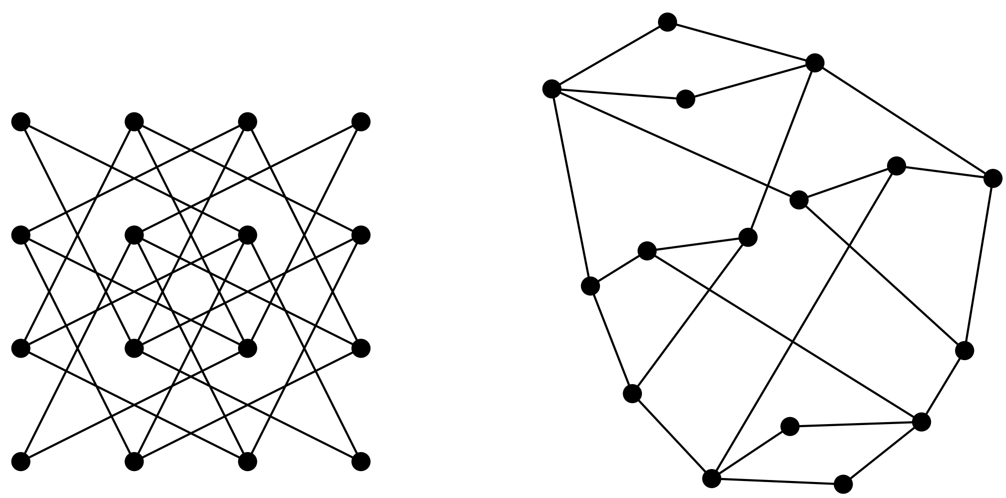

Figure 2: The graphs of Question 7

8. Prove that the graph in the right side of Figure 2 is not Hamiltonian.

9. Prove that the 4×4 chessboard does not admit a knight's tour.

10. Find a knight's tour on an 8×8 chessboard.

Hint. There are many knight's tours on the standard chessboard. One way to proceed (invented by our old acquaintance Leonard Euler) is to first find an open knight's tour on a 4×8 board that starts and ends at the top row; to find a knight's tour on the 8×8 board then you just "glue" that open tour and its mirror image.

11. List all possible trees with six or less vertices.

12. Draw the game tree for the game of nim if the starting position consists of three piles with one, two and three stones respectively. Which player has a winning strategy?

13. How many children does the root of the game tree for nim have if the starting position consists of three piles with seven, five and three stones respectively?

14. Give a solution to the Eight-Queens puzzle.