Midterm for CSI35

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Answer **three** of the following four questions. Be sure to indicate *clearly* which three questions you are answering.

1. Prove that for all natural numbers n we have:

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1}{6}$$

2. Prove that for all positive integers n, 7 divides $n^7 - n$.

3. Prove that for natural numbers $n \geq 7$ we have $3^n < n!$

4. Consider the following zero-one matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

   Prove that $A^n = A$ for all natural numbers $n \geq 1$, where the power is with respect to the boolean product.

Answer **one** of the following two questions. Be sure to indicate *clearly* which question you are answering.

5. Alice and Bob play a game by taking turns removing up to 4 stones from a pile that initially has n stones. The person that removes the last stone wins the game. Alice plays always first. For which values of n does Alice have a winning strategy? For which values of n does Bob have a winning strategy? Prove your answer.

6. In Nevereverland chicken nuggets come in packages of 3 and 5. Prove that for $n \geq 8$ a Nevereverlander can combine packages to get a total of exactly n chicken nuggets.

Answer **one** of the following two questions. Be sure to indicate *clearly* which question you are answering.

7. Let g_n be the number of bitstrings of length n with no consecutive ones. Give a recursive formula for g_n and prove your answer.

8. For an integer n let c_n be the number of ways that n can be written as a sum of ones, twos, threes, or fours where the order that the summands are written is important. Find a recursive definition of c_n and prove your answer.

Answer **one** of the following two questions. Be sure to indicate *clearly* which question you are answering.

9. A Morse code is a word in the alphabet consisting of two letters, the dot "·" and the dash "–". The two letters have different length, the dot has length 1 while the dash has length 2.

   (a) Give a recursive definition of the set of Morse codes M.

   (b) Give a recursive definition of the length $l(s)$ of a Morse code s.

   (c) Give a recursive formula for the number of Morse codes of length n. Prove this recursive formula.

10. On the set Σ^* of words from the alphabet $\Sigma = \{I, M, W\}$ define the flip $F(s)$ of a word s as follows:

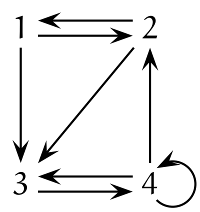
    - $F(\emptyset) = \emptyset$, where $\emptyset$ is the empty word
    - For a word s, $F(sI) = F(s)I$, $F(sW) = F(s)M$, and $F(sM) = F(s)W$

    Call a word *flippant* if $F(s) = R(s)$, where $R(s)$ stands for the reverse of s. For example, MIW is a flippant word.

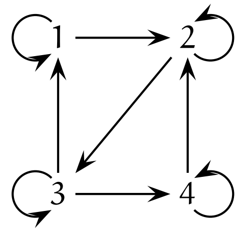
    (a) Give a recursive definition for the set of flippant words.

    (b) How many flippant words of length n are there? Give a formula and prove it.

Answer **both** of the following questions. These questions refer to the digraphs G_1 and G_2 shown bellow:



G_1



G_2

11. Answer the following questions for $i = 1, 2$:
    (a) Is G_i reflexive?
    (b) Is G_i irreflexive?
    (c) Is G_i symmetric?
    (d) Is G_i transitive?
12. Draw the digraph $G_2 \circ G_1$.

The following two questions are for **extra credit.** Do only one of them (if at all).

13. What is the last digit of 2009^{2009}?
14. Prove that 7 divides $5555^{2222} + 2222^{5555}$.