Answers to the Midterm for CSI35

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1. Prove that for all natural numbers n we have:

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1}{6}$$

Answer. We proceed by induction. For $n = 0$ we have the statement

$$\sum_{i=0}^{0} i^2 = \frac{0(0+1)(2 \cdot 0 + 1}{6}$$

which is true since both sides are evidently 0. This concludes the basic step.

For the inductive step, we assume that the sentence has been proven for n and we will prove it for $n+1$. That is, we assume that,

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

and we will prove that

$$\sum_{i=0}^{n+1} i^2 = \frac{(n+1)(n+1+1)\left(2(n+1)+1\right)}{6} \tag{1}$$

We start with the LHS of (1):

$$\begin{aligned}
\sum_{i=0}^{n+1} i^2 &= \sum_{i=0}^{n} i^2 + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\
&= \frac{(n+1)\left(n(2n+1) + 6(n+1)\right)}{6} \\
&= \frac{(n+1)\left(n(2n+1) + 6(n+1)\right)}{6} \\
&= \frac{(n+1)\left(2n^2 + 7n + 6\right)}{6}
\end{aligned}$$

On the other hand, the RHS of (1) is:

$$\begin{aligned}\frac{(n+1)(n+1+1)\left(2(n+1)+1\right)}{6} &= \frac{(n+1)(n+2)(2n+3)}{6} \\ &= \frac{(n+1)(2n^2+7n+6)}{6}\end{aligned}$$

Therefore the two sides of (1) are equal. This concludes the proof of (1) and the inductive step. □

2. Prove that for all positive integers n, 7 divides $n^7 - n$.

   Answer. We proceed by induction. The proposition is true when $n = 1$ since 7 divides 0. For the inductive step, we assume that 7 divides $n^7 - n$, that is we assume that $n^7 - n = 7k$ for some natural number k, and we are going to prove that 7 divides $(n+1)^7 - (n+1)$.

   We use the binomial theorem to expand $(n+1)^7$. We have:

$$(n+1)^7 = n^7 + 7n^6 + 21n^5 + 35n^4 + 35n^3 + 21n^2 + 7n + 1$$

   where the coefficients were computed as follows:

$$\binom{7}{0} = \binom{7}{7} = 1$$

$$\binom{7}{1} = \binom{7}{6} = 7$$

$$\binom{7}{2} = \binom{7}{5} = \frac{7 \cdot 6}{2!} = 21$$

$$\binom{7}{3} = \binom{7}{4} = \frac{7 \cdot 6 \cdot 5}{3!} = 35$$

   So we have:

$$\begin{aligned}(n+1)^7 - (n-1) &= n^7 + 7n^6 + 21n^5 + 35n^4 + 35n^3 + 21n^2 + 7n + 1 - n - 1 \\ &= n^7 - n + 7(n^6 + 3n^5 + 5n^4 + 5n^3 + 3n^2 + n) \\ &= 7k + 7(n^6 + 3n^5 + 5n^4 + 5n^3 + 3n^2 + n) \\ &= 7(k + n^6 + 3n^5 + 5n^4 + 5n^3 + 3n^2 + n)\end{aligned}$$

   So 7 divides $(n+1)^7 - (n+1)$, and this concludes the inductive step and the proof. □

3. Prove that for natural numbers $n \geq 7$ we have $3^n < n!$

Answer. By induction. For the basic step $n = 7$, we have: $3^7 = 2187$ while $7! = 5040$. Therefore, $3^7 < 7!$. This completes the basic step.

For the inductive step we assume that

$$3^n < n! \tag{2}$$

and we are going to prove that $3^{n+1} < (n+1)!$ Indeed, since $n \geq 7$ we have that

$$3 < n + 1 \tag{3}$$

Multiplying the inequalities (2) and (3) we get, since all terms involved are positive:

$$3^n \cdot 3 < n! \cdot (n + 1)$$

Or equivalently,

$$3^{n+1} < (n + 1)!$$

This completes the inductive step and the proof. □

4. Consider the following zero-one matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Prove that $A^n = A$ for all natural numbers $n \geq 1$, where the power is with respect to the boolean product.

Answer. The proposition is obviously true for $n = 1$. For the inductive step we assume

$$A^n = A$$

and we'll prove that

$$A^{n+1} = A$$

We have:

$$\begin{aligned}
A^{n+1} &= A^n \cdot A \\
&= A \cdot A \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \\
&= A
\end{aligned}$$

This conclusions the inductive step and the proof. □

5. Alice and Bob play a game by taking turns removing up to 4 stones from a pile that initially has n stones. The person that removes the last stone wins the game. Alice plays always first. For which values of n does Alice have a winning strategy? For which values of n does Bob have a winning strategy? Prove your answer.

   Answer. We will prove that Bob has a winning strategy when $n = 5k$, for some natural number k. For all other values of n Alice has a winning strategy.

   To prove that for $n = 5k$ Bob has a winning strategy, we proceed by induction. For $k = 1$, that is when there are $n = 5$ stones, Bob can win no matter what is Alice's first move. Indeed, if Alice takes 1, Bob can take the remaining 4 stones and win, if Alice takes 2, Bob can take the remaining 3 stones and win, if Alice takes 3, Bob can take the remaining 2 stones and win, and if Alice takes 4, Bob can take the remaining stone and win.

   For the inductive step, we assume that Bob has a winning strategy when there are $n = 5k$ stones, and we'll prove that he has a strategy when $n = 5(k+1)$ as well. Now, $5(k+1) = 5k+5$ so no matter what Alice's first move is, Bob can ensure that after his second move there are $5k$ stones left. (Indeed, if Alice takes 1 stone Bob takes 4, if Alice takes 2 Bob takes 3, if Alice takes 3 Bob takes 2 and when Alice takes 4 Bob takes 1; this way after Bob's first move, 5 stones have been removed in total, leaving $5k$ stones.) Once there are $5k$ stones with Alice's turn to play, Bob can follow the strategy guaranteed by the inductive hypothesis and win. This completes the inductive step.

   Now if n is not a multiple of 5, it will leave remainder 1, 2, 3 or 4 when divided by 5. In other words, for some k we'll have $n = 5k+1$, or $n = 5k+2$, or $n = 5k+3$, or

$n = 5k + 4$; so in her first move Alice can take 1, or 2, or 3, or 4 respectively leaving $5k$ stones with Bob's turn to play. We proved in the previous paragraph that when there $5k$ stones the second player has a winning strategy. So once there are $5k$ stones and Bob's turn to play, Alice can follow that strategy and win. □

6. In Nevereverland chicken nuggets come in packages of 3 and 5. Prove that for $n \geq 8$ a Nevereverlander can combine packages to get a total of exactly n chicken nuggets.

   Answer. We will use strong induction. The basic step is $n = 8$: We can get 8 nuggets by using a package of three and a package of 5. This completes the basic step.

   For the indcutive step, we will assume that the statement is true for all integers k with $8 \leq k < n$ and we will prove that it it is also true for n. In other words we will assume that one can combine packages to get any number of nuggets k with $8 \leq k < n$ and prove that one can also get n nuggets by combining packages.

   Indeed, if n is large enough so that $n - 3 \geq 8$, i.e. if $n \geq 11$, then by the inductive hypothesis we can get $n - 3$ nuggets by combining packages, so we can get n nuggets by just adding a package of 3. If n is not greater of equal than 11 it will be 8, 9, or 10. We've already seen that we can get 8 nuggets. We can also get 9 by using 3 packages of 3 and we can get 10 nuggets by using 2 packages of 5. So we can get any number $n \geq 8$ and this completes the inductive step and the proof. □

7. Let g_n be the number of bitstrings of length n with no consecutive ones. Give a recursive formula for g_n and prove your answer.

   Answer. The recursive formula is:

$$g_n = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ g_{n-1} + g_{n-2} & \text{if } n > 1 \end{cases}$$

   This definition is correct for $n = 0$ because the empty bitstring has no consecutive ones in it so we have one bitstring of length 0 with no consecutive ones. When $n = 2$ we have two bitstrings 0, and 1 and none of them has consecutive ones. So $g_1 = 2$.

   To make descriptions brief, let us call a bitsring *good* if it has no consecutive ones. For $n > 1$, the set of good bitstrings length n is the union of two disjoint sets: those that start with 0 and those that start with 1.

   We can take any good bitstring of length $n - 1$ and preppend 0 to it to get a good bitstring of length n starting with 0, therefore there are at least g_{n-1} good bitstrings of length n starting with 0. On the other hand there are at most g_{n-1} good bitstrings of length n starting with 0, since if we delete the initial 0 from such a bitstring we get a good bitstring of length $n-1$. It follows that there are exactly g_{n-1} good bistrings of length n that start with 0.

If we take any good bitstring of length $n-2$ and no consecutive ones and prepend 10 to it we get a good bitstring of length n starting wiht 1. So there are at least g_{n-2} good bistrings of length n starting with 1. On the other hand, a good bitstring of length n that starts with 1 actually has to start with 10, so we can delete the initial two bits of any such bitstring to get a good bitstring of length $n-2$. Thus there are at most g_{n-2} good bitstrings of length n starting with 1. It follows that there are exactly g_{n-2} good bitsrings of length n that start with 1.

So we have, $g_n = g_{n-1} + g_{n-2}$ as needed. □

8. For a positive integer n let c_n be the number of ways that n can be written as a sum of ones, twos, threes, or fours where the order that the summands are written is important. Find a recursive definition of c_n and prove your answer.

Answer. The recursive formula is:

$$c_n = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ 4 & \text{if } n = 3 \\ 8 & \text{if } n = 4 \\ c_{n-1} + c_{n-2} + c_{n-3} + c_{n-4} & \text{if } n > 4 \end{cases}$$

The only way we can write 1 as a sum using 1, 2, 3 and 4 is simply

$$1 = 1$$

Therefore $c_1 = 1$.

For $n = 2$ we have two ways:

$$2 = 1 + 1, \quad \text{or} \quad 2 = 2$$

Therefore $c_2 = 2$.

For $n = 3$ we have:

$$3 = 1 + 1 + 1, \quad \text{or} \quad 3 = 1 + 2, \quad \text{or} \quad 3 = 2 + 1, \quad \text{or} \quad 3 = 3$$

so $c_3 = 4$

For $n = 4$ we have:

$4 = 1 + 1 + 1 + 1$, or $4 = 1 + 1 + 2$, or $4 = 1 + 2 + 1$, or $4 = 1 + 3$, or $4 = 2 + 1 + 1$, or $4 = 2 + 2$ or $4 = 3 + 1$, or $4 = 4$

so $c_4 = 8$.

For $n > 4$ the set of ways that we can write n as a sum of ones, twos, threes, or fours is the union of four disjoint sets: those that start with 1, those that start with 2, those that srart with 3 and those that start with 4.

Now, there are c_{n-1} ways to write $n-1$ as a sum of ones, twos, threes, or fours and by adding 1 at the beginning of such a way we get a way to write n as a sum of ones, twos, threes, or fours. So there are at least c_{n-1} ways to write n as a sum of ones, twos, threes, or fours that have 1 as first summand. On the other hand if we have a sum of ones, twos, threes, or fours with first summand 1 and total sum of n, we can delete the first summand and get a sum of ones, twos, threes, or fours that adds up to $n-1$. So there are at most c_{n-1} ways to write n as a sum of ones, twos, threes, or fours that have 1 as first summand. If follows that there are at most c_{n-1} ways to write n as a sum of ones, twos, threes, or fours that have 1 as first summand.

By entirely analogous arguments we can prove that there are c_{n-2} ways that start with 2, c_{n-3} ways that start with 3 and c_{n-4} ways that start with 4.

Therefore there are $c_{n-1} + c_{n-2} + c_{n-3} + c_{n-4}$ ways to write n as a sum of ones, twos, threes, or fours. □

9. A Morse code is a word in the alphabet consisting of two letters, the dot "·" and the dash "—". The two letters have different length, the dot has length 1 while the dash has length 2.

   (a) Give a recursive definition of the set of Morse codes M.

   (b) Give a recursive definition of the length $l(s)$ of a Morse code s.

   (c) Give a recursive formula for the number of Morse codes of length n. Prove this recursive formula.

Answer. (a) The set of Morse codes is recursively defined as follows:

- The empty code $\emptyset$ is a Morse code.
- If m is a Morse code so are $m\cdot$ and $m-$.
- All Morse codes are generated by the previous two rules.

(b) Since $\cdot$ has length 1 and $-$ has length 2, every time we append a dot at the end of a Morse code the length increases by one while every time that we append a dash the length increases by 2. So we have the following recursive definition:

- $l(\emptyset) = 0$
- $l(m\cdot) = l(m) + 1$
- $l(m-) = l(m) + 2$

(c) If F_n is the number of Morse codes of length n, we have the following formula:

$$F_n = \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2 \end{cases}$$

This definition is correct for $n = 0$ because there is only one Morse code of length 0, namely the empty code. It is also correct for $n = 1$ since there is only one Morse code of length 1, namely "$\cdot$".

For $n \geq 2$ we have that the set of Morse codes of length n is the union of two disjoint sets: those that end with $\cdot$ and those that end with $-$.

There are F_{n-1} Morse codes of length n that end with $\cdot$. To see that there are at least F_{n-1} codes of length n that end in $\cdot$, notice that we can append a dot to any Morse code of length $n-1$ to get a code of length n that ends with a dot. To see that there are at most F_{n-1} codes of length n that end in a dot, notice that if we remove the last dot from such a code we get a code of lenth $n-1$.

Similarly, there are F_{n-2} Morse codes of length n that end with $-$. For, appending a dash to a code of length $n-2$ yields a code of length n ending in $-$, while removing the final dash from a code of length n that ends in a dash yields a code of length n.

Therefore, there are $F_{n-1} + F_{n-2}$ Morse codes of length n. □

10. On the set Σ^* of words from the alphabet $\Sigma = \{I, M, W\}$ define the flip $F(s)$ of a word s as follows:

- $F(\emptyset) = \emptyset$, where $\emptyset$ is the empty word
- For a word s, $F(sI) = F(s)I$, $F(sW) = F(s)M$, and $F(sM) = F(s)W$

Call a word *flippant* if $F(s) = R(s)$, where $R(s)$ stands for the reverse of s. For example, MIW is a flippant word.

(a) Give a recursive definition for the set of flippant words.

(b) How many flippant words of length n are there? Give a formula and prove it.

Answer. (a) The following is a recursive definition of the set of fillpant words:

- The empty word is a flippant. The words of length 1, I, M and W are flippant.
- If s is a flippant word then so are MsW, WsM and IsI.
- All flippant words are generated by the previous rules.

(b) If f_n is the number of the flippant words of length n, we have the following formula:

$$f_n = \begin{cases} 3^{\frac{n}{2}} & \text{if } n \text{ is even} \\ 3^{\frac{n+1}{2}} & \text{if } n \text{ is odd} \end{cases} \tag{4}$$

Using the ceiling function we can express the above in a single formula:

$$f_n = 3^{\lceil \frac{n}{2} \rceil}$$

To prove this we first prove the following recursive formula:

$$f_n = \begin{cases} 1 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 3f_{n-2} & \text{if } n \geq 2 \end{cases} \tag{5}$$

This formula true for $n = 1$ and $n = 2$ as follows from the basic step of the recursive definition of the set of flippant words.

If $n \geq 2$, there are at least $3f_{n-2}$ flippant words, since for each flippant word of length $n - 2$ the recursive step of the definition in part a gives three different flippant words of length n. To see that there cannot be more than $3f_{n-2}$ flippant words of length n, notice that if a flippant word of length n starts with M it has to end with W, if it starts with W it has to end with M and if it starts with I it has to end with I. So all flippants words of length n come from a flippant word of length $n - 2$ by prepending M and appending W, or by prepending W and appending M or by prepending and appending I. Thus there are exactly $3f_{n-2}$ flippant words of length n.

Using formula (5) and strong induction we can prove formula (4) as follows:

The formula (4) is true for $n = 1$ and $n = 2$ as seen by simply substituting. Now for the inductive step assume that the formula is true for all numbers less than n and we will prove that it is also true for n. Now according to formula (5)

$$f_n = 3f_{n-2}$$

If n is even, by the inductive hypothesis $f_{n-2} = 3^{\frac{n-2}{2}}$ since $n - 2$ is also even. So for n even we have:

$$\begin{aligned} f_n &= 3f_{n-2} \\ &= 3 \cdot 3^{\frac{n-2}{2}} \\ &= 3^{\frac{n-2}{2}+1} \\ &= 3^{\frac{n}{2}} \end{aligned}$$

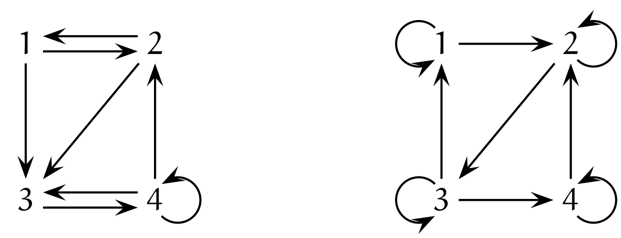
If n is odd then $f_{n-2} = 3^{\frac{n-2+1}{2}}$ since $n-2$ is also odd. Therefore:

$$\begin{aligned} f_n &= 3f_{n-2} \\ &= 3 \cdot 3^{\frac{n-2+1}{2}} \\ &= 3^{\frac{n-1}{2}+1} \\ &= 3^{\frac{n+1}{2}} \end{aligned}$$

This completes the inductive step and the proof.

□

The following questions refer to the digraphs G_1 and G_2 shown bellow:



G_1 G_2

11. Answer the following questions for $i = 1, 2$:
    (a) Is G_i reflexive?
    (b) Is G_i irreflexive?
    (c) Is G_i symmetric?
    (d) Is G_i transitive?

Answer. (a) G_1 is not reflexive, G_2 is.

(b) Neither G_1 nor G_2 are irreflexive.

(c) Neither G_1 nor G_2 are symmetric.

(d) Neither graph is transitive. To see this we will work with the matrices associated with the digraphs. The matrix associated with G_1 is:

$$M_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Taking the boolean square of M_1 we have

$$M_1 \odot M_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \odot \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Since, for example, the $(1, 1)$ element of $M_1 \odot M_2$ is 1 while the $(1, 1)$ element of M_1 is 0 we conclude that M_1 is not transitive.

The matrix associated with G_1 is:

$$M_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Taking the boolean square of M_2 we have

$$M_1 \odot M_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Since, for example, the $(1, 3)$ element of $M_2 \odot M_2$ is 1 while the $(1, 3)$ element of M_2 is 0 we conclude that M_2 is not transitive.

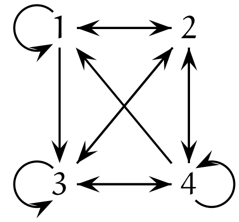
□

12. Draw the digraph $G_2 \circ G_1$.

Answer. We first calculate the matrix associated with $G_2 \circ G_1$.

$$M_2 \odot M_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \odot \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

So we have the following graph:



$G_2 \circ G_1$

□

13. What is the last digit of 2009^{2009}?

Answer. The last digit of 2009^{2009} is 9. To see this we first observe that for any natural number n the last digit of 2009^n will be the same as the last digit of 9^n, this follows from the well known algorithm for multiplication. We will prove by induction that

The last digit of 9^n is 1 if n is even and 9 if n is odd.

Indeed this is true for $n = 0$ since 0 is even and $9^0 = 1$. Assume then that it is true that the last digit of 9^n is 1 or 9 according to whether n is even or odd. We will prove that this is true for the last digit of 9^{n+1} as well.

If $n+1$ is odd then n is even and according to the inductive hypothesis the last digit of 9^n is 1. Now $9^{n+1} = 9^n \cdot 9$ and when we use the standard multiplication algorithm we'll get 9 to be the last digit of 9^{n+1}. Similarly, if $n+1$ is odd, then n is even and therefore the last digit of 9^n is 9. So when we use the standard multiplication algorithm to multiply 9^n with 9 we'll get 1 as the last digit. □

14. Prove that 7 divides $5555^{2222} + 2222^{5555}$.

Answer. Since 5555 leaves remainder 4 when divided by 7 and 2222 leaves remainder 3 when divided by 7, it follows that $5555^{2222} + 2222^{5555}$ leaves the same remainder as $4^{2222} + 3^{5555}$ when divided by 7. Now

$$\begin{aligned}
4^{2222} + 3^{5555} &= 4^{2222} + 3^{5555} \\
&= 4^{2\cdot 1111} + 3^{5\cdot 1111} \\
&= \left(4^2\right)^{1111} + \left(3^5\right)^{1111} \\
&= (16)^{1111} + (243)^{1111} \\
&= (16+243)(16^{1110} - 16^{1109}243 + \cdots - 16\cdot 243^{1109} + 243^{1110}) \\
&= 259(16^{1110} - 16^{1109}243 + \cdots - 16\cdot 243^{1109} + 243^{1110}) \\
&= 7\cdot 37(16^{1110} - 16^{1109}243 + \cdots - 16\cdot 243^{1109} + 243^{1110})
\end{aligned}$$

□

Thus $4^{2222} + 3^{5555}$ is divisible by 7 and therefore so is $5555^{2222} + 2222^{5555}$.