Systems of two linear equations with two unknown

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Recall. If one draws two lines on the plane there are three possibilities:

1. The two lines intersect in exactly one point.
2. The two lines are parallel.
3. The two lines coincide (they are drawn one on top of the other).

In the previous lecture we saw that if we know the equations of the two lines then we can decide which of the three possibilities we have. Specifically, assuming that none of the lines is vertical, we can write the equations in the *slope-intercept* form and then we have the following three possibilities respectively:

1. The two lines have different slopes.
2. The two lines have the same slope but their intercepts are different.
3. The two lines have the same equation: the same slope and the same intercept.

If one of the two lines is vertical then the three possibilities are:

1. The other line is not vertical.
2. The other line is vertical but different, i.e. it has different x-intercept.
3. The other line is vertical and has the same x-intercept.

The question we want to answer now is: "given the equations of two lines, find out whether they intersect or not, *and* in case that they intersect find (the coordinates of) their common points". In some cases this may be straightforward, for example:

Example 1. Consider the lines $L_1 : x = -3$ and $L_2 : y = 1$. Find their common points.

Answer. The first line contains all points with first coordinate -3. The second line contains all points with second coordinate 1. There is only one point that satisfies both of these conditions namely the point with coordinates $(-3, 1)$. □

Now you answer the following questions:

1. Find the common points the two lines $L_1 : y = 9$ and $L_2 : x = 4$.

2. Consider the lines with equations $x = 5$ and $y = 0$. Find their intersection point.

The previous examples were straightforward, in essence they illustrated the very definition of the Cartesian coordinate system; remember the streets and avenues? Even if only one of the lines is a "coordinate line" (i.e. a vertical or horizontal line) we can find the intersection point rather easily:

Example 2. Find the common point of the two lines: $2x - 3y = 7$ and $x = -2$.

Answer. The second line contains all points with x-coordinate equal to -2. So we need to find the y–coordinate of the point of the first line that has x-coordinate -2. We know how to do that: we substitute $x = -2$ in the first equation and solve for y.

$$\begin{aligned} 2(-2) - 3y = 7 &\iff -4 - 3y = 7 \\ &\iff -3y = 7 + 4 \\ &\iff -3y = 11 \\ &\iff y = -\frac{11}{3} \end{aligned}$$

So the common point has coordinates $\left(-2, -\frac{11}{3}\right)$. □

Example 3. Find the common point of the two lines: $3x + 2y = -8$ and $y = \frac{3}{4}$.

Answer. We need to find the point of the first line that has $y = \frac{3}{4}$. We substitute:

$$\begin{aligned} 3x + 2\left(\frac{3}{4}\right) = -8 &\iff 3x + \frac{3}{2} = -8 \\ &\iff 3x = -\frac{3}{2} - 8 \\ &\iff 3x = -\frac{3}{2} - \frac{16}{2} \\ &\iff 3x = -\frac{19}{2} \\ &\iff x = -\frac{19}{6} \end{aligned}$$

So the common point is $\left(-\frac{19}{6}, \frac{3}{4}\right)$. □

Example 4. Find the common points of the lines with equations $3x + 7y = -2$ and $2y = -4$.

Answer. The second line is horizontal (since its equation does not involve x) but is is not solved for y. So we first have to solve the equation for y:

$$2y = -4 \iff y = -2$$

Now we can substitute in the equation of the first line:

$$3x + 7(-2) = -2$$

Solving this equation gives $x = 4$. So the intersection point is $(4, -2)$. $\square$

Lets practice a bit:

1. Find the intersection point of the lines with equations $x = 8$ and $4x - 3y = 32$.

2. Find the common point of the lines $y = -3$ and $-3x + 2y = 5$.

3. Find the common points of the lines with equations $2x + 4y = 7$ and $3x = 6$.

This idea of getting information from one equation and plugging it to the other equation can be used in general. We will not develop that method[1] in full; instead we will show some examples of how these ideas can be used to find the intersection of two more general lines:

Example 5. Find the point where the lines with equations $2x-3y=-9$ and $y=2x-1$, intersect.

Answer. At the point of intersection both of these equations will be true. So we can take the information that the second equation gives us, namely that y is equal to $2x-1$, and substitute it in the first equation, that is we replace every occurrence of y with $2x-1$. The first equation then becomes

$$2x-3(2x-1)=9$$

The last equation has only one variable x and we can solve it:

$$\begin{aligned} 2x-3(2x-1)=9 &\iff 2x-6x+3=-9 \\ &\iff -4x+3=-9 \\ &\iff -4x=-12 \\ &\iff x=3 \end{aligned}$$

So the x–coordinate of the intersection point is 3. Substituting this into the second equation we obtain

$$y=2(3)-1$$

or equivalently

$$y=5$$

So the intersection point is $(3,5)$. $\square$

Example 6. Find the point where the lines with equations $y=3x-1$ and $x-y=4$ intersect.

Answer. We substitute the information given in the first equation into the second and solve the resulting one-variable equation.

$$\begin{aligned} x-(3x-1)=4 &\iff x-3x+1=4 \\ &\iff -2x+1=4 \\ &\iff -2x=3 \\ &\iff x=-\frac{3}{2} \end{aligned}$$

Now that we know the x-coordinate of the intersection point we substitute it in the second equation to find the y–coordinate of the intersection point as well:

$$y=3\left(-\frac{3}{2}\right)-1 \iff y=-\frac{11}{2}$$

Thus the intersection point has coordinates $\left(-\frac{3}{2},-\frac{11}{2}\right)$. $\square$

[1] This method is called the *substitution method.*

Lets practice:

1. Find the point that the line with equation $y = -2x + 5$ and the line with equation $3x + 4y = 15$ have in common.

2. Find the point of intersection of the lines with equation $x = 2y - 3$ and $5x - 3y = 6$.

Systems

In this part we will study the algebraic counterpart of finding the common points for a pair of lines.

When we use the phrase "a *system* of equations" we mean two or more equations considered *at the same time*: we are interested to find solutions *common* to all of the equations. In this class we will concentrate on systems of two linear equations with two unknowns. In other words, we'll have two linear equations, each equation will have two variables[2], and we want to find their common solutions, the ordered pairs that satisfy both equations simultaneously. To indicate that we consider the two equations together as a system we use a left brace, like this:

$$\begin{cases} x - 5y &= -28 \\ 3x + 7y &= 26 \end{cases}$$

To solve such a system, then means to find all common solution of the two equations. So a *solution* of a system of two equations with two unknowns is an ordered pair of numbers that is a solutions to both equations of the system.

Geometrically, each of the two equations represents a line, and finding the common solutions is finding the point of intersection (if any). So for a system of two linear equations with two unknowns there are three possibilities:

1. The two lines intersect at exactly one point. In that case there is only one solution to the system.

2. The two lines are parallel. In that case the system has no solutions. A system with no solutions is called *inconsistent*.

3. The two lines are really the same. In that case we have infinitely many solutions: any solution to one of the equations is a solution to the system. Such a system is called *indeterminate*.

In the previous section we show glimpses of a method for finding the intersection point of two lines. The method outlined there is called the *substitution method*. In this section we will develop an other method for solving systems of equations, the *elimination method*[3].

Given a system of equations, we can perform the following two operations to the system and the system we obtain has the same exactly solutions as the one we started with. The first two operations we have already seen when solving equations with one variable.

- **Operation O:** Transfer terms (with opposite sign) from one side of the equation to the other.

- **Operation A:** Multiply both sides of one of the equations with the same number.

- **Operation B:** Add the two equations together and replace one of the two equations with this "added up" equation. Of course, by "adding the two equations" we mean adding the expressions on the RHS, adding the expressions on the LHS, and setting the two sums equal.

The *elimination method* for solving linear systems uses these three operations to gradually transform the original system into a system whose solutions are obvious. We arrive to this final stage by *eliminating* one variable from each equation in turn. Before stating the method step by step, lets see some examples of its use:

[2] The *same* two variables.

[3] In the literature this method is sometimes called the *addition method*.

Example 7. Solve the following system:

$$\begin{cases} -3x + 5y &= 12 \\ 3x + 7y &= 24 \end{cases}$$

Answer. We add the two equations together and we replace the first equation with the "added up" equation. The point of adding the equations up, is that the variable x has opposite coefficients so it will be *eliminated* from the result. Indeed adding the two equations we get:

$$12y = 36$$

So the new system is

$$\begin{cases} 12y &= 36 \\ 3x + 7y &= 24 \end{cases}$$

Now we can multiply the first equation with $\frac{1}{12}$ to get:

$$\begin{cases} y &= 3 \\ 3x + 7y &= 24 \end{cases}$$

Now multiply the first equation with -7 and add it to the second. The point of this is to eliminate y from the second equation. So we get:

$$\begin{cases} -7y &= -21 \\ 3x + 7y &= 24 \end{cases}$$

and then

$$\begin{cases} -7y &= -21 \\ 3x &= 3 \end{cases}$$

The final step is to divide the first equation by -7 and the second by 3:

$$\begin{cases} y &= 3 \\ x &= 1 \end{cases}$$

The solution to the last system is obvious: $(1, 3)$. This is the solution of the original system also. $\square$

Example 8. Solve the following system:

$$\begin{cases} 2x - 5y &= -2 \\ 3x + 5y &= -3 \end{cases}$$

Answer. Now we notice that the variable y has opposite coefficients in the two equations. So if we add the two equations the "added up" equation won't have a y in it, in other words we will *eliminate* y. The "added up" equation is:

$$5x = -5$$

Replacing the first equation of the system with the "added up" equation we get:

$$\begin{cases} -5x &= -5 \\ 3x + 5y &= -3 \end{cases}$$

Divide the first equation by -5 to get:

$$\begin{cases} x & = -1 \\ 3x + 5y & = -3 \end{cases}$$

Now we can use the first equation to eliminate x from the second. To do that multiply the first equation with -3, the *opposite* of the coefficient of x in the second equation. The first equation then becomes:

$$-3x = 3$$

After adding this to the second equation we get

$$5y = 0$$

so the system becomes:

$$\begin{cases} -3x & = 3 \\ \quad 5y & = 0 \end{cases}$$

Now divide the first equation by -3 and the second by 5 to get:

$$\begin{cases} x & = -1 \\ \quad y & = 0 \end{cases}$$

So the solution of the system is $(-1, 0)$. $\square$

In the previous two examples in the original system there was a variable (x in the first and y in the second) that was "ready for elimination", i.e. its coefficients on the two equations were opposite. This won't always be the case. In general we will need to use Operation A, in order to have a variable ready for elimination.

Example 9. Solve the system:

$$\begin{cases} x + 4y & = 14 \\ 3x + 2y & = 12 \end{cases}$$

Answer. In this system none of the variables is ready to be eliminated. Notice that if we multiply the first equation by -3 the two coefficients of the variable x will be opposite. So the system becomes:

$$\begin{cases} -3x - 12y & = -42 \\ 3x + 2y & = 12 \end{cases}$$

We then add the two equations and replace the second equation by the "added up" equation:

$$\begin{cases} -3x - 12y & = -42 \\ \quad - 10y & = -30 \end{cases}$$

We now divide the second equation by -10. In order to simplify the system we also divide the first equation by -3 [4]:

$$\begin{cases} x + 4y & = 14 \\ \quad y & = 3 \end{cases}$$

[4] Remember that two steps earlier we had multiplied the first equation with -3 in order to eventually eliminate the x from the second equation. Now that we have accomplished that we can bring the first equation back to the simpler form.

Now we can multiply the second equation with -4 to get $-4y = -12$. Adding this equation to the first we get the system:

$$\begin{cases} x & = 2 \\ \quad y & = 3 \end{cases}$$

So the solution is $(2, 3)$. $\square$

Before giving more examples lets formalize the content of the last footnote. Often we use operation A in order to prepare one variable for elimination, that is we multiply the equation with a number so that one of the variables ends up with opposite coefficients; we then proceed to add the two equations together eliminating that variable. After this elimination process is over we are left with a multiplied equation which is usually more "complicated" than the original "unmultiplied" one. Of course, as we have been doing, we can always divide the multiplied equation by the number we multiplied it with to get back the original simpler equation. From now on, after the elimination has been accomplished we will automatically revert the multiplied equation to its original form without explicit mention of the division we have to perform. Now back to examples:

Example 10. Solve the following system:

$$\begin{cases} -3x + 2y & = -2 \\ 5x + 4y & = 7 \end{cases}$$

Answer. Notice that the coefficient of y in the second equation is a multiple of the coefficient of y in the first equation. So we can multiply the first equation by a number (namely -2) to make y ready for elimination. So multiplying the first equation by -2 we get

$$6x - 4y = 4$$

Adding this equation to the second equation we get:

$$\begin{cases} -3x + 2y & = -2 \\ 11x & = 11 \end{cases}$$

Dividing the second equation by 11 we get

$$\begin{cases} -3x + 2y & = -2 \\ x & = 1 \end{cases}$$

Now we multiply the second equation with 3 to get

$$3x = 3$$

Now add this to the first equation to get

$$\begin{cases} 2y & = 1 \\ x & = 1 \end{cases}$$

Now divide the first equation by 2 to get:

$$\begin{cases} y & = \frac{1}{2} \\ x & = 1 \end{cases}$$

So the solution is $\left(\frac{1}{2}, 1\right)$. $\square$

In the most general case we will have to multiply both equations with suitable numbers in order to make one variable ready for elimination.

Example 11. Solve the system:

$$\begin{cases} 3x+2y &= 2 \\ 5x+3y &= 4 \end{cases}$$

Answer. We will eliminate the variable y first. In order to do this we will multiply the first equation with 3 (the coefficient of y in the second equation) and the second equation with -2 (the opposite of the coefficient of y in the first equation). Then we get

$$\begin{cases} 9x+6y &= 6 \\ -10x-6y &= -8 \end{cases}$$

Now we can add these two equation and replace the second equation of the original system with the added up equation:

$$\begin{cases} 3x+2y &= 2 \\ -x &= -2 \end{cases}$$

Now multiply the second equation with -1:

$$\begin{cases} 3x+2y &= 2 \\ x &= 2 \end{cases}$$

Now we can multiply the second equation with -3 to get $-3x=-6$ and add this equation to the first to get:

$$\begin{cases} 2y &= -4 \\ x &= 2 \end{cases}$$

Finally divide the first equation by 2 to get

$$\begin{cases} y &= -2 \\ x &= 2 \end{cases}$$

So the solution is $(2,-2)$. □

In general to solve a linear system of two equation and two unknowns using the *elimination method* we proceed as follows:

0. Separate knowns and unknowns in both equations, by transferring all variable terms to the LHS, and all constant terms to the RHS.

1. Multiply one or both equations with suitable numbers so that one of the variables is ready for elimination. If there is no obvious easier way, this step can always be accomplished by choosing one variable and then multiplying the first equation with the coefficient of that variable in the second equation, and the second equation with the opposite of the coefficient of that variable in the first equation[5].

[5]or vice versa: we multiply the first equation with the opposite of the second coefficient and the second equation with the first coefficient. The important thing is that when we "cross-multiply" we keep one coefficient as is and we change the sign of the other.

2. Add the two new equations and replace one of the equations in the *original* system with the "added up" equation. Notice that the variable chosen in the previous step has been eliminated in the "added up" equation.

3. Divide the coefficient of the remaining variable in the added up equation to get this equation in "solved" form.

4. Multiply the solved equation with the opposite of coefficient of its remaining variable in the other equation and add this multiplied equation to the other equation. This eliminates the the variable not chosen in the second step from the other equation.

5. Divide the other equation by the coefficient of the variable chosen in the second step.

Lets see some more worked out examples.

Example 12. Solve the following system:

$$\begin{cases} -3x &= y - 2 \\ 5x + 2y &= 5 \end{cases}$$

Answer. The first equation is not separated. So we first separate knowns and unknowns in the first equation.

$$\begin{cases} -3x - y &= -2 \\ 5x + 2y &= 5 \end{cases}$$

We will first eliminate y. We multiply the first equation by 2 and add it to the second equation.

$$\begin{cases} -3x - y &= -2 \\ -\ x &= 1 \end{cases}$$

Multiply both equations with -1 to get:

$$\begin{cases} 3x + y &= 2 \\ x &= -1 \end{cases}$$

Now multiply the second equation with -3 and add it to the first to get:

$$\begin{cases} y &= 5 \\ x &= -1 \end{cases}$$

So the solution is $(-1, 5)$. □

Example 13. Solve the system:

$$\begin{cases} 3x + 2y &= 5 \\ 9x + 6y &= 8 \end{cases}$$

Answer. The coefficient of x in the second equation is 3 times the coefficient of x in the first. So lets multiply the first equation with -3 and add it to the second, to eliminate x. Now something "funny" happens, both variables are eliminated from the second equation:

$$\begin{cases} 3x + 2y &= 5 \\ 0 &= -7 \end{cases}$$

The second equation is impossible (a contradiction). Therefore this system is inconsistent: it has no solutions. □

Example 14. Solve the system:

$$\begin{cases} 3x + 2y & = 5 \\ 9x + 6y & = 15 \end{cases}$$

Answer. The coefficient of x in the second equation is 3 times the coefficient of x in the first. So lets multiply the first equation with -3 and add it to the second, to eliminate x. Again something "funny" happens, both variables are eliminated from the second equation:

$$\begin{cases} 3x + 2y & = 5 \\ 0 & = 0 \end{cases}$$

Now however, the second equation is a tautology. Therefore this system is indeterminate, it is really equivalent to the first equation. So the solutions to this system are all solutions of the equation $3x + 2y = 5$. $\square$

Now lets practice:

1. Solve the system:

$$\begin{cases} 3x + 2y & = 6 \\ 6x - 2y & = 0 \end{cases}$$

2. Solve the system:

$$\begin{cases} 5x + y &= 9 \\ 2x + 3y &= 14 \end{cases}$$

3. Solve the system

$$\begin{cases} 3x - 2y &= 8 \\ 9x - 6y &= 14 \end{cases}$$

4. Solve the system

$$\begin{cases} 4x - 5y &= -1 \\ 10x + 5y &= 14 \end{cases}$$

5. Solve the system:

$$\begin{cases} -3x - 7y &= 17 \\ -2x + 5y &= -37 \end{cases}$$

Mixing the methods

In the previous sections we show two different methods for solving a linear system: the substitution method (which we didn't really develop in full) and the elimination method. In practice, we quite often mix the two methods: we use the elimination method to eliminate one variable from one equation but after finding the value of one of the variables we substitute in the other equation to find the other variable. This method is illustrated in the following example.

Example 15. Solve the system:

$$\begin{cases} -3x + 2y &= 19 \\ \quad 4x + 6y &= 18 \end{cases}$$

Answer. We'll first eliminate y: multiply the first equation with -3 and add it to the second.

$$\begin{cases} -3x + 2y &= \quad 19 \\ \quad 13x &= -39 \end{cases}$$

Now divide the second equation by 13:

$$\begin{cases} -3x + 2y &= 19 \\ \qquad x &= -3 \end{cases}$$

Now this system can be easily solved using the substitution method: just substitute -3 for x in the first equation to get:

$$-3(-3) + 2y = 19$$

Solving this equation[6] gives:

$$y = 5$$

So the solution is $(-3, 5)$. □

[6] How? do it!