Practice for the second exam

1. Evaluate: $8 - 4(2-3) - 2^2 \div 4 \cdot 5.$

   Solution. We have:

$$\begin{aligned} 8 - 4(2-3) - 2^2 \div 4 \cdot 5 &= 8 - 4(-1) - 4 \div 4 \cdot 5 \\ &= 8 + 4 - 1 \cdot 5 \\ &= 8 + 4 - 5 \\ &= 7 \end{aligned}$$

   □

2. Evaluate $2a - b$ when $a = 3$ and $b = -3$.

   Solution. We have:

$$\begin{aligned} 2a - b &= 2(3) - (-3) \\ &= 6 + 3 \\ &= 9 \end{aligned}$$

   □

3. Solve the equation:

$$2(3x-2) + x = 5x - 4$$

   Solution. We have:

$$\begin{aligned} 2(3x-2) + x = 5x - 4 &\iff 6x - 4 + x = 5x - 4 \\ &\iff 7x - 4 = 5x - 4 \\ &\iff 7x - 5x = 4 - 4 \\ &\iff 2x = 0 \\ &\iff x = 0 \end{aligned}$$

   □

4. Solve the equation:

$$|5x - 10| = 20$$

   Solution. We have

$$|5x - 10| = 20 \iff 5x - 10 = 20 \text{ or } 5x - 10 = -20$$

   We have to solve each of these equations:

$$\begin{aligned} 5x - 10 = 20 &\iff 5x = 30 \\ &\iff x = 6 \end{aligned}$$

while,

$$\begin{aligned} 5x - 10 = -20 &\iff 5x = -10 \\ &\iff x = -2 \end{aligned}$$

So we have two solutions: -2 and 6. $\square$

5. Solve the following inequality, give the answer using interval notation and graph the solution set.

$$3 - 2x \leq 6 - 3(3x + 8)$$

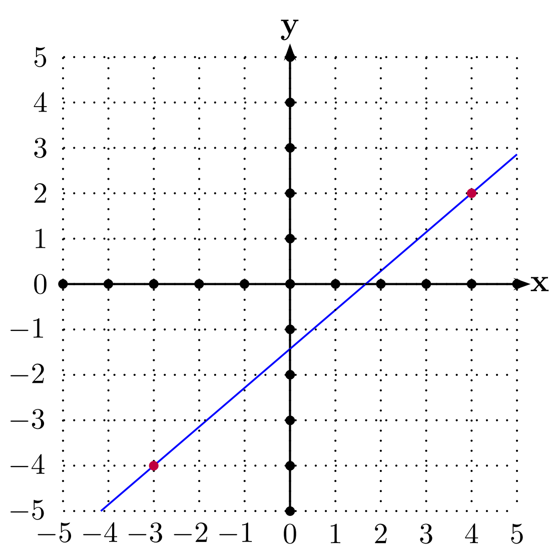
Solution. We have:

$$\begin{aligned} 3 - 2x \leq 6 - 3(3x + 8) &\iff 3 - 2x \leq 6 - 9x - 24 \\ &\iff 3 - 2x \leq -9x - 18 \\ &\iff -2x + 9x \leq -3 - 18 \\ &\iff 7x \leq -21 \\ &\iff x \leq -3 \end{aligned}$$

So the solution set is $(-\infty, -3]$. The graph of the solution set is:

$\square$

6. Find the equation of the line whose graph is shown. Then find its x– and y–intercepts.



Solution. This line passes through the points with coordinates $(-3, -4)$ and $(4, 2)$. Therefore its slope is

$$\frac{\Delta y}{\Delta x} = \frac{2-(-4)}{4-(-3)} = \frac{6}{7}$$

So the point-slope form of the equation of this line will be

$$y = \frac{6}{7}x + b$$

where b is the y–intercept. Since the line passes through the point with coordinates $(4, 2)$ we will have:

$$\begin{aligned} 2 = \frac{6}{7} \cdot 4 + b &\Longleftrightarrow 2 = \frac{24}{7} + b \\ &\Longleftrightarrow 2 - \frac{24}{7} = b \\ &\Longleftrightarrow -\frac{10}{7} = b \end{aligned}$$

So the equation of the line is

$$y = \frac{6}{7}x - \frac{10}{7}$$

We've already seen that the y-intercept of this line is $-\frac{10}{7}$. To find the x–intercept of the line we substitute $y = 0$ in its equation and solve for x:

$$\begin{aligned} 0 = \frac{6}{7}x - \frac{10}{7} &\Longleftrightarrow 0 = 6x - 10 \\ &\Longleftrightarrow 10 = 6x \\ &\Longleftrightarrow \frac{10}{6} = x \\ &\Longleftrightarrow \frac{5}{3} = x \end{aligned}$$

So the x–intercept of the line is $\frac{5}{3}$. □

7. Find the equation of the line that is parallel to the line $y = -3x + 5$ and has the same y–intercept as the line $2x + 3y = 12$.

   Solution. The line has the same slope as the line $y = -3x + 5$. Therefore the slope is -3. We also know that the intercept will be the same as the intercept of the line $2x + 3y = 12$. Substituting $x = 0$ in the last equation gives $3y = 12$ or equivalently, $y = 4$. Therefore the y–intercept is 4. In sum the line has slope -3 and y–intercept 4. It follows that the equation of the line is

$$y = -3x + 4$$

□

8. Find the intersection point of the following lines: $3x + y = -3$ and $x - y = -5$

   Solution. We need to solve the system:

$$\begin{cases} 3x + y &= -3 \\ \quad x - y &= 5 \end{cases}$$

   We use the elimination method: we add the two equations and replace the first equation with the summed up equation:

$$\begin{cases} 4x &= 2 \\ x - y &= 5 \end{cases}$$

   Solving the first of these equations gives:

$$x = \frac{1}{2}$$

   Substituting this value into the second equation we get:

$$\frac{1}{2} - y = 5$$

   Solving this equation gives:

$$y = -\frac{9}{2}$$

   So the two lines intersect at the point with coordinates $\left(\frac{1}{2}, -\frac{9}{2}\right)$. $\square$

9. Solve the following system:

$$\begin{cases} 4x - 3y &= -3 \\ 3x + 2y &= 19 \end{cases}$$

   Solution. We'll solve the system using the method of elimination: We start by eliminating y: We multiply the first equation by 2 and the second by 3.

$$\begin{cases} 8x - 6y &= -6 \\ 9x + 6y &= 57 \end{cases}$$

   We then add the two equations and replace the second equation with the "added up" equation. We also revert the first equation to its original form:

$$\begin{cases} 4x - 3y &= -3 \\ 17x &= 51 \end{cases}$$

   Now divide the second equation by 17:

$$\begin{cases} 4x - 3y &= -3 \\ \quad x &= 3 \end{cases}$$

Now multiply the second equation with -4, add it to the first and replace the first equation with the "added up" equation:

$$\begin{cases} \quad - 3y &= -15 \\ x &= \quad 3 \end{cases}$$

Finally, divide the first equation by -3:

$$\begin{cases} \quad y &= 5 \\ x &= 3 \end{cases}$$

So the solution is $(3, 5)$. □

10. Solve the following system:

$$\begin{cases} x - 3y &= 5 \\ 2x - 6y &= 8 \end{cases}$$

Solution. We want to eliminate x. So we multiply the first equation with -2 and add it to the second. We get:

$$0 = -2$$

Since this is a contradiction, we conclude that the system is inconsistent: there are no solutions. □

11. John has \$1.15 all in dimes and nickels. He has a total of 15 coins. How many of each kind of coin does he have?

Solution. Let x be the number of nickels and y the number of dimes. Then all the nickels together are worth $5x$ cents and all the dimes together are worth $10y$ cents. So all the coins are worth $5x + 10y$ cents. So the first equation we get is

$$5x + 10y = 115$$

Since there are 15 coins altogether, we also have

$$x + y = 15$$

Thus we have to solve the system:

$$\begin{cases} 5x + 10y &= 115 \\ x + \quad y &= \quad 15 \end{cases}$$

Multiply the second equation by -5, add it to the first and replace the first equation with the "added up" equation:

$$\begin{cases} \quad 5y &= 40 \\ x + y &= \quad 15 \end{cases}$$

Solving the first equation we get

$$y = 8$$

Substituting this value to the second equation then gives:

$$x + 8 = 15$$

Solving gives

$$x = 7$$

So John has seven nickels and eight dimes. □

12. Simplify:

$$\left(\frac{2x^3y^5z^2}{-3x^4y^3z^4}\right)^3 (6x^4y^5z^6)^2$$

Solution. We have:

$$\begin{aligned}
\left(\frac{2x^3y^5z^2}{-3x^4y^3z^4}\right)^3 (6x^4y^5z^6)^2 &= \left(\frac{2y^2}{-3xz^2}\right)^3 \cdot 36x^8y^{10}z^{12} \\
&= \frac{8y^6}{-27x^3z^6} \cdot 36x^8y^{10}z^{12} \\
&= \frac{8y^6}{-3} \cdot 4x^5y^{10}z^6 \\
&= -\frac{32x^5y^{16}z^6}{3}
\end{aligned}$$

□

13. Simplify: $\dfrac{30a^5b^3 - 25a^3b^2 - 10a^4b^6 + 5a^2b}{5a^2b}$

Solution. We have:

$$\begin{aligned}
\frac{30a^5b^3 - 25a^3b^2 - 10a^4b^6 + 5a^2b}{5a^2b} &= \frac{30a^5b^3}{5a^2b} - \frac{25a^3b^2}{5a^2b} - \frac{10a^4b^6}{5a^2b} + \frac{5a^2b}{5a^2b} \\
&= 6a^3b^2 - 5ab - 2a^2b^5 + 1
\end{aligned}$$

□

14. Divide: $\dfrac{3x^2 - 5x - 3}{x - 5}$

Solution. We perform long division:

$$
\begin{array}{r|l}
 & \quad 3x + 10 \\
\cline{2-2}
x - 5 & \quad 3x^2 - 5x - 3 \\
3x^2 - 15x & -3x^2 + 15x \\
\cline{2-2}
 & \qquad 10x - 3 \\
10x - 50 & \quad -10x + 50 \\
\cline{2-2}
 & \qquad\quad 47
\end{array}
$$

So we have:

$$\frac{3x^2 - 5x - 3}{x - 5} = 3x + 10 + \frac{47}{x - 5}$$

□

15. Multiply: $(2x - 3)^3$

Solution. We have:

$$
\begin{aligned}
(2x - 3)^3 &= (2x - 3)(2x - 3)^2 \\
&= (2x - 3)(4x^2 - 12x + 9) \\
&= 8x^3 - 24x^2 + 18x - 12x^2 + 36x - 27 \\
&= 8x^3 - 36x^2 + 54x - 27
\end{aligned}
$$

□

16. Factor completely: $10x^3y^3 + 4xy - 15x^2y^2 - 6$

Solution. We'll factor by grouping: the first two terms have a common factor of $2xy$ and the last two terms have a common factor of -3.

$$
\begin{aligned}
10x^3y^3 + 4xy - 15x^2y^2 - 6 &= 2xy(5x^2y^2 + 2) - 3(5x^2y^2 + 2) \\
&= (5x^2y^2 + 2)(2xy - 3)
\end{aligned}
$$

□

17. Factor: $27x^3 - 8y^3$

Solution. This is the difference of two cubes: the cube of $3x$ minus the cube of $2y$. So using the "difference of cubes" identity we have:

$$
\begin{aligned}
27x^3 - 8y^3 &= (3x - 2y)\left((3x)^2 + (3x)(2y) + (2y)^2\right) \\
&= (3x - 2y)\left(9x^2 + 6xy + 4y^2\right)
\end{aligned}
$$

□

18. Solve: $4x^3 + 8x^2 - 9x - 18 = 0$

Solution. This is a higher degree equation. We'll solve it by factoring the LHS and then use the zero factor property. We factor the LHS by grouping the first two terms and the last two terms:

$$\begin{aligned} 4x^3 + 8x^2 - 9x - 18 = 0 &\iff 4x^2(x+2) - 9(x+2) = 0 \\ &\iff (x+2)(4x^2 - 9) = 0 \\ &\iff (x+2)(2x+3)(2x-3) = 0 \\ &\iff x + 2 = 0 \text{ or } 2x + 3 = 0 \text{ or } 2x - 3 = 0 \\ &\iff x = -2 \text{ or } x = -\frac{3}{2} \text{ or } x = \frac{3}{2} \end{aligned}$$

□

19. Solve: $\quad 3x^2 - 8x = 0$

Solution. We have:

$$\begin{aligned} 3x^2 - 8x = 0 &\iff x(3x - 8) = 0 \\ &\iff x = 0 \text{ or } 3x - 8 = 0 \\ &\iff x = 0 \text{ or } x = \frac{8}{3} \end{aligned}$$

□

20. Solve: $\quad x^2 - 7x - 44 = 0$

Solution. To solve the equation we need to factor the LHS, quadratic trinomial with $a = 1$, $b = -7$ and $c = -44$. So we look for two numbers whose product is -44 and whose sum is -7. The two numbers turn out to be -11 and 4. So we have:

$$\begin{aligned} x^2 - 7x - 44 = 0 &\iff x^2 - 11x + 4x - 44 = 0 \\ &\iff x(x - 11) + 4(x - 11) = 0 \\ &\iff (x - 11)(x + 4) = 0 \\ &\iff x = 11 \text{ or } x = -4 \end{aligned}$$

□

21. Solve: $\quad x^4 - 5x^2 - 36 = 0$

Solution. This is an fourth degree equation without cubic or linear terms. Such polynomials can be factored with methods similar to those used for quadratic equations. In our case, we look for two numbers whose products is -36 and whose sum is -5. The numbers are -9 and 4. So we have:

$$\begin{aligned}
x^4 - 5x^2 - 36 = 0 &\Leftrightarrow x^4 - 9x^2 + 4x^2 - 36 = 0 \\
&\Leftrightarrow x^2(x^2 - 9) + 4(x^2 - 9) = 0 \\
&\Leftrightarrow x^2(x^2 - 9) + 4(x^2 - 9) = 0 \\
&\Leftrightarrow (x^2 - 9)(X^2 + 4) = 0 \\
&\Leftrightarrow (x + 3)(x - 3)(X^2 + 4) = 0 \\
&\Leftrightarrow x + 3 = 0 \text{ or } x - 3 = 0 \text{ or } X^2 + 4 = 0 \\
&\Leftrightarrow x = -3 \text{ or } x = 3
\end{aligned}$$

Where in the last line we disregarded the $x^2 + 4 = 0$ part since $x^2 + 4$ can never be 0.[1] □

22. Solve: $6x^2 = x + 2$

    Solution. We first transfer all the terms of the equation to the LHS and then factor and solve.

    $$6x^2 = x + 2 \Leftrightarrow 6x^2 - x - 2 = 0$$

    So we have to find two numbers with product -12 and with sum -1. The numbers are -4 and 3. So we have:

    $$\begin{aligned}
    6x^2 - x - 2 = 0 &\Longleftrightarrow 6x^2 - 4x + 3x - 2 = 0 \\
    &\Longleftrightarrow 2x(3x - 2) + 3x - 2 = 0 \\
    &\Longleftrightarrow (3x - 2)(2x + 1) = 0 \\
    &\Longleftrightarrow 3x - 2 = 0 \text{ or } 2x + 1 = 0 \\
    &\Longleftrightarrow x = \frac{2}{3} \text{ or } x = -\frac{1}{2}
    \end{aligned}$$

    □

23. Find a number such that six times the square of the number is equal to the sum of that number and two.

    Solution. Let x be the number. Then we have

    $$6x^2 = x + 2$$

    This equation was solved in Question 22. So there are two such numbers $\frac{2}{3}$ and $-\frac{1}{2}$. □

24. The product of two consecutive even numbers is 48. Find the two numbers.

[1] Why not?

Solution. Let x be the smaller of the two numbers, then the other number will be $x+2$. So we have the equation:

$$x(x+2) = 48$$

We solve:

$$\begin{aligned} x(x+2) = 48 &\Longleftrightarrow x^2 + 2x = 48 \\ &\Longleftrightarrow x^2 + 2x - 48 = 0 \\ &\Longleftrightarrow x^2 + 8x - 6x - 48 = 0 \\ &\Longleftrightarrow x(x+8) - 6(x+8) = 0 \\ &\Longleftrightarrow (x+8)(x-6) = 0 \\ &\Longleftrightarrow x = -8 \text{ or } x = 6 \end{aligned}$$

So there are two pairs of such numbers, -8 and -6 is the first pair, 6 and 8 the second. □

25. The length of a rectangle is 2 cm less than 3 times its width. If the area of the rectangle is $40\,\mathrm{cm}^2$ find its perimeter.

Solution. Let x be the width of the rectangle in centimeters. Then its length will be $3x-2$. Then the area of the rectangle will be

$$x(3x-2) = 3x^2 - 2x$$

So to find the dimensions of the rectangle we need to solve the equation

$$3x^2 - 2x = 40$$

We have:

$$\begin{aligned} 3x^2 - 2x = 48 &\Leftrightarrow 3x^2 - 2x - 40 = 0 \\ &\Leftrightarrow 3x^2 - 12x + 10x - 40 = 0 \\ &\Leftrightarrow 3x(x-4) + 10(x-4) = 0 \\ &\Leftrightarrow (x-4)(3x+10) = 0 \\ &\Leftrightarrow x - 4 = 0 \text{ or } 3x + 10 = 0 \\ &\Leftrightarrow x = 4 \text{ or } x = -\frac{10}{3} \end{aligned}$$

Since the width cannot be negative we conclude that the width of the rectangle is 4 cm. Then the length will be $3 \cdot 4 - 2 = 10$ cm. So the perimeter will be $2 \cdot 4 + 2 \cdot 10 = 28$ cm. □