Polynomials and operations among them

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November 3, 2008

Recall. Towards the beginning of this class we saw:

- A *variable* is a symbol that represents an unknown or varying number. Variables are usually letters of the Latin alphabet but sometimes letters from other alphabets are used as well.
- An *algebraic expression* is made up by applying some operations (e.g. addition, subtraction, multiplication, division, exponentiation, absolute value, square root, ...) to variables and numbers.
- An algebraic expression can be *evaluated* once we assign a value to all of the variables that occur in it[1]. The evaluation is done by replacing all occurrences of each variable with its assigned value and then perform the indicated operations.

In this part of the class we will concentrate on *polynomials.* Polynomials are algebraic expressions made up by applying addition, subtraction and/or multiplication to variables and numbers. Notice that division is missing from the above list of operations. More precisely, polynomials are the expressions that can be built using the following rules

1. Numbers are polynomials. When we think of numbers as polynomial we often call them *constant polynomials* or simply *constants.*
2. All variables are polynomials.
3. The opposite of a polynomial is a polynomial.
4. If we multiply two polynomials the result is again a polynomial.
5. If we add or subtract two polynomials the result is again a polynomial.

Now let's see some examples.

Example 1. Each of the following expressions is a polynomial:

$$3, \quad y, \quad x^2, \quad -2x^2y, \quad 2-x^2z^3, \quad \frac{2}{5}x^3-5x+3x^2-7, \quad -2x(x^3-5x)$$

Indeed, the first one is a number and therefore a polynomial. The second one is a variable and therefore a polynomial. The third one is a polynomial because x^2 is the result of multiplying x with x. The fourth one is the result of multiplying the polynomials -2, x, x and y. The fifth one is the difference of the polynomials 2 and x^2z^3. The sixth one is a polynomial because it is the sum of four polynomials $\frac{2}{5}x^3$, $-5x$, $3x^2$, and -7. Finally the last expression is polynomial because it is the product of the polynomials $-2x$ and x^3-5x.

[1]Sometimes we assign values to *only some* of the variables. In that case we have *partial evaluation.*

Example 2. None of the following expressions is a polynomial:

$$\frac{3}{x}, \quad \sqrt{4x}, \quad |x+4|, \quad 2^x$$

For, the first one involves division, the second one involves square roots, the third one involves absolute value and the last one involves exponentiation.

At this point we should note the difference between expressions that involve division by numbers and expressions that involve division with more general expressions. The sixth expression in Example 1 is a polynomial even though there is division involved (namely by the number 5). The reason is that to divide by a number means to multiply by its inverse number, so to divide by 5, say, is OK because we are really multiplying with $\frac{1}{5}$, a number. However when we divide by x, say, we are multiplying with $\frac{1}{x}$, which is not a polynomial.

We should also note the difference between constant and variable exponents. The third expression in Example 1 is a polynomial because x^2 is really $x \cdot x$ so the expression really involves only multiplication of variables. In general when the exponent is a positive natural number then the expression only involves multiplication and therefore it is OK. When we have a variable exponent however the exponent is not necessarily a natural number so we cannot say that the expression involves only multiplication. For example in the expression 2^x the variable x may be assigned the value -5 or even $\frac{2}{7}$ and at this point we have no idea what 2^{-2} or $2^{2/7}$ mean, in particular it is not clear that only multiplication is involved[2].

Let's practice:

1. Circle all polynomials:
   - A. $-2x^2yz^3$
   - B. $\dfrac{3x^2-5x+3}{3x^2}$
   - C. $\dfrac{7xy^3-2z}{7}$
   - D. $|x^2-2y^2|$
   - E. x^3-5x^2+7x-3
   - F. x^3-3^x

Terminology

If a polynomial involves only multiplication of numbers and variables then it is called a *monomial* or a *term*. So for example the following are monomials:

$$3x, \quad -5, \quad 5xyz^3, \quad 7x^4, \quad -6x^3, \quad x^2y^3. \quad \frac{2x^5y}{5}, \quad -x^3$$

A term is a product of a number and some variables. The number is called the *coefficient* of the term and the product of the variables is called the *variable part* of the term. If a term has is a constant term (i.e. just a number) then by convention we consider its variable part to be 1. If there

[2] At a later time we will give meaning to these expressions. We will see then than more than multiplication is involved in, say, $2^{2/7}$.

is no coefficient present then we can assume that the coefficient is 1. So in the terms listed above the coefficients are

$$3, \quad -5, \quad 5, \quad 7, \quad -6, \quad 1. \quad \frac{2}{5}, \quad -1$$

and the variable parts are

$$x, \quad 1, \quad xyz^3, \quad x^4, \quad x^3, \quad x^2y^3. \quad x^5y, \quad x^3$$

respectively.

If two terms have the same variable part then they are called *like* terms. So for example $-3x^2y$ and $5x^2y$ is a pair of like terms while $6x^2y$ and $3xy^2$ are not like terms. How about the terms $-3x^2y$ and $5yx^2$, are they like terms? Well, strictly speaking their variable parts are not *exactly* the same, the first has variable part x^2y and the second has variable part yx^2, so we might think that they are not like terms. However, since if we change the order of the factors the product doesn't change, no matter what the values of we give to x and y when we evaluate x^2y and yx^2 we will get the same result. So, even though they are *written* differently we consider x^2y and yx^2 to be the same, or as we say in mathematics *equal*. For this reason, the terms $-3x^2y$ and $5yx^2$ *are* like terms after all. In sum,

> Two terms are considered like when they have the same variables in the same powers. The order that the variables are written is not important.

A polynomial that is the sum of two terms is called a *binomial*. So for example the following are binomials:

$$3x - 4, \quad 5x^2y^3 - 2x^3y^4, \quad 3x^3 - 2x$$

A polynomial that is the sum of three terms is called a *trinomial*. So for example the following are trinomials:

$$2x^2 - 5x + 3, \quad -2x^4 - 5x^3 + 3x, \quad xy^3 - 5x^2yz - 7z^5$$

In general as we shall see, all polynomials can be written as a sum of unlike terms. When a polynomial is written like that we say that it is in *simplified expanded form*. So all binomials and trinomials in the previous examples are in simplified expanded form. The following two polynomials are *not* in simplified expanded form:

$$-2x(3x^3 - 4x), \quad 2x^2 + 3x - 5x + 2$$

The first is not a sum of terms but a product of a term and a binomial. The second is a sum of terms (so it is in expanded form) however two of the terms are like, namely $3x$ and $-5x$, so it's not on simplified expanded form. Most of the times we want to have polynomials in simplified expanded form. So let's state this as a rule[3]:

> Unless explicitly told not to, we should write all polynomials in simplified expanded form.

[3]We will see an exception to this rule in the next chapter when we will discuss *factoring*.

In the most part of this chapter we will deal with the question "How can we get a polynomial into simplified expanded form?"

The number of variables that are multiplied to give the variable part of a term is called the *degree* of that term. In other words the degree of a term is the sum of the exponents of all the variable of the term. So for example, the degree of $3x^4y^5$ is 9, the degree of $-3x^3$ is 3, the degree of $-2x$ is 1, the degree of $-5xy^2z^3$ is 6, and the degree of 5 is 0. In general the degree of all constant terms, *except* 0, is 0. The degree of the constant term 0 is *undefined*[4].

More generally, the degree of a polynomial in simplified expanded form, is the highest degree of all of its terms. So for example, $2x-3$ has degree 1, $7xy-3x+6$ has degree 2, and $-7x^5+6x^4+x-1$ has degree 5.

A polynomial is written in *ascending order* if it is in simplified expanded form and the terms are listed in order of their degree, starting with the terms of lowest degree and ending with the terms of highest degree. So for example the following polynomials are written in ascending order:

$$4-2x+3x^2, \quad 5x-x^4, \quad 2+3x-4x^2+5x^3-7x^4$$

A polynomial can also be written in *descending order*: its terms are listed in order according to their degree, starting with the highest degree and ending with the lowest. For example the following are written in descending order:

$$3x^2-2x+4, \quad -x^4+5x, \quad -7x^4+5x^3-4x^2+3x+2$$

In general we will be writing the polynomials in descending order.

We conclude with some more commonly used terminology: A polynomial of degree 1 is called *linear*. So the following are linear polynomials:

$$2x-5, \quad 3x-5y+5, \quad 3x, \quad 3x-5y-z-3$$

A polynomial of degree 2 is called *quadratic*. The following are quadratic polynomials:

$$x^2, \quad 2y^2-5, \quad 3x-3xy, \quad x^2+7x, \quad 2x^2-3x+6$$

A polynomial of degree 3 is called *cubic*. The following are cubic polynomials:

$$3x^3-5x, \quad x^2y-5x^2-y^2+4x-3, \quad 7x^3-2x^2+3x-5$$

Let's practice all this terminology:

1. Give an example of
   (a) a linear polynomial with 3 variables.

[4] The reason is that 0 when considered as a polynomial can have *any* variable part. All of the following are equal to the zero polynomial: $0, \quad 0x^2, \quad 0x^2y^6z^5w$

(b) A quadratic trinomial with one variable.

(c) A cubic binomial of with one variable.

(d) A cubic polynomial with one variable whose quadratic term has coefficient -7.

(e) A polynomial with one variable, of degree 5 whose cubic term has coefficient -7, and its linear term has coefficient 1.

(f) A quadratic trinomial whose quadratic term has coefficient -1, its linear term has coefficient 5, and its constant term is -9.

Evaluating polynomials

Polynomials are algebraic expressions and they can therefore be evaluated. In this section we review evaluation and develop some special notation for it. Let's start with some practice:

1. Evaluate $x^2 - y^2$ when:

   A. $x = 5$, $y = 4$.

   B. $x = 2$, $y = -2$.

2. Evaluate $(x+y)(x-y)$ when:
   A. $x=5$, $y=4$.

   B. $x=2$, $y=-2$.

3. Evaluate $(x-3)^2$ when:
   A. $x=5$.

   B. $x=-3$.

4. Evaluate x^2-9 when:
   A. $x=5$.

   B. $x=-3$.

5. What is the moral of the above calculations?

Often we will use names for polynomials so that we can refer to the same polynomial again and again without having to write it down in detail every time. We will use letters as names of polynomials, and will indicate in parenthesis what are the variables of the polynomial. So if we want a name for a polynomial in one variable x we may use the names $p(x)$, $q(x)$, $a(x)$, etc. If we want a name of a polynomial of two variables x and y we may use $p(x, y)$, $q(x, y)$, etc.

With this notation if we use $p(x)$ as the name of a polynomial, then we will use $p(-2)$ to mean the result of evaluating $p(x)$ when $x = -2$, $p(6)$ to mean the result of evaluating $p(x)$ when $x = 6$, etc. Let's practice this new notation:

1. Let $p(x) = -2x^2 - 3x$. Find
   (a) $p(0)$

   (b) $p(1)$

   (c) $p(-3)$

2. Let $p(x) = (x - 3)^2$ and $q(x) = x^2 - 6x + 9$. Find
   (a) $p(0)$ and $q(0)$.

   (b) $p(5)$ and $q(5)$.

   (c) $p(-3)$ and $q(-3)$.