Lines and their equations

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Recall. In the previous lecture we saw:

1. The *slope-intercept* form of the equation of a line is

$$y = mx + b \tag{1}$$

   where m and b are real numbers.

2. A line with equation (1) has slope m and crosses the y–axis at the point $(0, b)$. Conversely, a line with slope m and y–intercept b has equation (1).

3. Every *non-vertical* line has an equation in the slope intercept form. In particular:

   - A *horizontal* line that crosses the y–axis at the point $(0, b)$ has equation

$$y = b \tag{2}$$

   - A *vertical* line does **not** have an equation in slope intercept form. If a vertical line crosses the x–axis at the point $(a, 0)$ then its equation is

$$x = a \tag{3}$$

Last time we also developed a method for finding the equation of a line given its slope and a point in it. We illustrate the method in the following example:

Example 1. Find the equation of the line that passes through the point with coordinates $(6, -5)$ and has slope $-\dfrac{2}{3}$.

Solution. Let b be the y–intercept of the line. Then its equation will be:

$$y = -\frac{2}{3}x + b$$

When a line passes through a point, the coordinates of the point satisfy the equation of the line. Since our line passes through the point $(6, -5)$ we must have:

$$-5 = -\frac{2}{3}(6) + b$$

or equivalently,

$$-5 = -4 + b$$

So that

$$b = -1$$

Therefore the equation of the line is

$$y = -\frac{2}{3}x - 1$$

$\square$

Lets practice:

1. Find an equation of the line with slope 3 and y–intercept -5.

2. Find an equation of the line that passes through the point $\left(0, -\frac{1}{2}\right)$ and has slope 4.

3. Find the equation of a line that has slope -2 and passes through the point $(-3, 3)$.

4. A line passes through the point $(4, -2)$ and has slope $\dfrac{2}{3}$. Find its equation.

5. A line passes through $(8, -3)$ and has slope 0. Find its equation.

6. A vertical line passes through $\left(1, -\dfrac{2}{5}\right)$. Find its equation.

7. A line has slope $-\frac{3}{2}$ and passes through the point $\left(\frac{4}{9}, \frac{1}{6}\right)$. Find an equation for this line.

8. A horizontal line passes through the point $(5, -2)$. Find its equation.

9. A line passes through the points $(1, 2)$ and $(4, 5)$.
   (a) Find the slope of this line.

   (b) Now that you know the slope of the line use one of the given points to find an equation for this line.

(c) Now use the *other* point to find an equation. Do you get the same equation?

10. Use the method outlined in the previous exercise to find the equation of the line that passes through the points with coordinates $(-1, 2)$ and $(-5, 6)$.

Which point did you choose to "plug in"? Why?

11. Find the equation of the line that passes through the points $(-1, 1)$ and $(2, 7)$.

12. Find the equation of the line that passes through the points $(5, 6)$ and $(-3, 2)$.

13. Find the equation of the line that passes through the points $(5, -2)$ and $(-2, -2)$.

14. Find the equation of the line that passes through the points $(3, 4)$ and $(3, -2)$.

Another method for finding equations of lines

There is an other method we can use to find the equation of a line if we know its slope and a point in it. We illustrate the method with an example.

Example 2. Find the equation of the line that has slope -3 and passes through the point $(-1, 5)$.

Solution. Let (x, y) be any point in this line different than the given point $(-1, 5)$. Then the line passes through the points $(-1, 5)$ and (x, y); therefore its slope is

$$\frac{\Delta y}{\Delta x} = \frac{y-5}{x-(-1)}$$

$$= \frac{y-5}{x+1}$$

We are given that the slope is -3 so we have the equation

$$\frac{y-5}{x+1} = -3 \tag{4}$$

Multiplying both sides with the denominator we get:

$$y - 5 = -3(x+1) \tag{5}$$

Now we can solve equation (5) for y to arrive at the slope-intercept form of the equation. To do this use the distributive property in the RHS, move -5 to the RHS where it becomes $+5$, and finally combine terms to get:

$$y = -3x + 2$$

□

The equation (5) is in *point-slope* form.[1]

[1] Sometimes in the literature equation (4) is also said to be in point-slope form.

Before proceeding we should confess that while solving Example 2 we swept some details under the rug. Notice that to arrive in equation (5) we had to assume that the point with coordinates (x, y) was different than the point $(-1, 5)$. However *a posteriori* we can verify that the point $(-1, 5)$ satisfies equation (5) also[2]. So equation (5) is satisfied by *all* points of the line.

Lets practice this method a bit:

1. A line passes through $(-2, -3)$ and has slope 6.
   (a) Use the method outlined in the previous example to find an equation of this line in point-slope form.

   (b) Solve the equation you arrived at for y to get a point-slope form of the equation of the line.

2. Use the point-slope method to find an equation for the line that passes through the point $(-1, 3)$ and has slope 2. Write the equation in slope intercept form.

[2] Verify it!

Lets work the general case. Assume that a line with slope m passes through the point with coordinates (a, b). Then let (x, y) be a different point in this line. We can compute the slope

$$\frac{\Delta y}{\Delta x} = \frac{y - b}{x - a}$$

$$= \frac{y - b}{x - a}$$

So we must have:

$$\frac{y - b}{x - a} = m$$

Multiplying both sides with $(x - a)$ we arrive at the equation:

$$y - b = m(x - a) \tag{6}$$

Notice that even though we assumed that $(x, y) \neq (a, b)$, if we substitute $x = a$ and $y = b$ in equation (6) we get

$$b - b = m(a - a)$$

which is equivalent to

$$0 = m \cdot 0$$

which is a true equation. Therefore *all* points of the line satisfy equation (6).

Notice that now we can use formula (6) to get the point-slope form of a line. For example:

Example 3. Find the equation of the line with slope 4 that passes through the point $(3, 5)$.

Solution. The point slope form of the equation of this line is:

$$y - 5 = 4(x - 3)$$

Solving this equation for y (how? do it!) we can get the equation in slope-intercept form:

$$y = 4x - 7$$

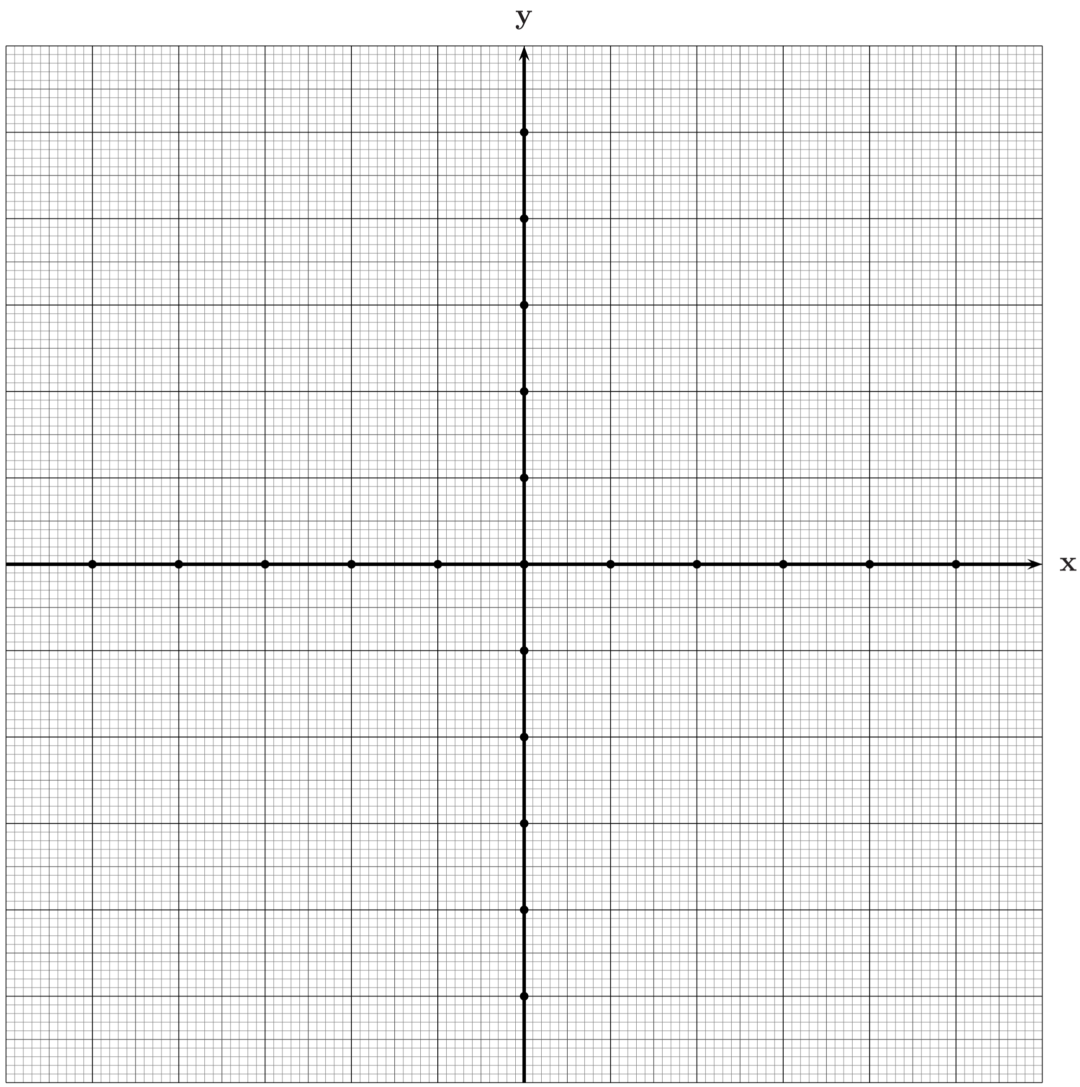
□

What the slope tells us

In this section we answer the question: "What we know about a line if we know its slope?" or in other words, "What do lines with equal slope have in common?"

Lets start by experimenting.

1. Consider the lines $L_1 : y = 2x - 1$, $L_2 : y = 2x$, and $L_3 : y = 2x + 1$. Graph all three lines in the same grid. What do you observe?

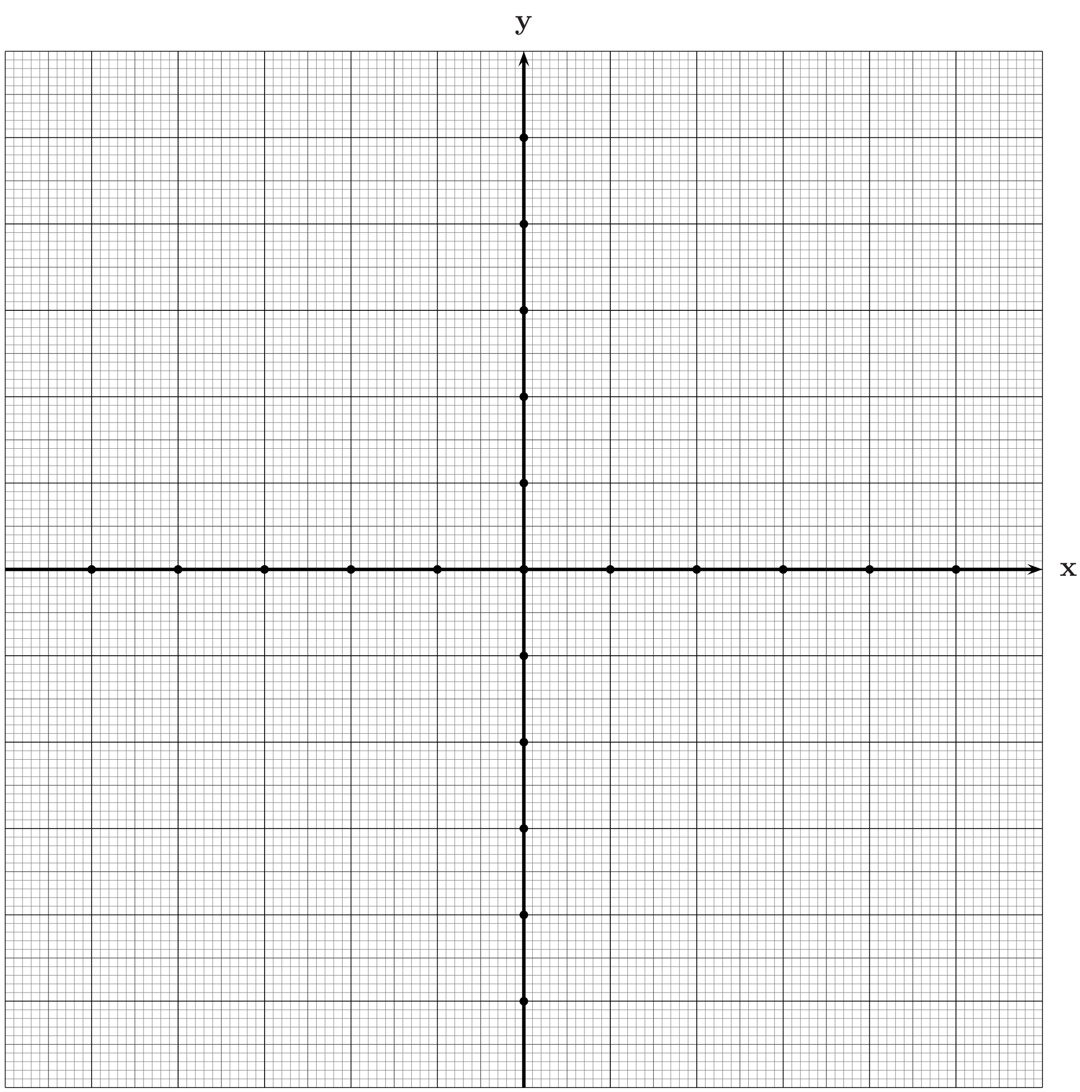


2. Write down four lines with slope 0. Is your previous observation confirmed?

3. Consider the lines $L_1 : 2x - 4y = 0$, $L_2 : 2x - 4y = 2$, and $L_3 : 2x - 4y = 4$.

   (a) Verify that all three lines have the same slope.

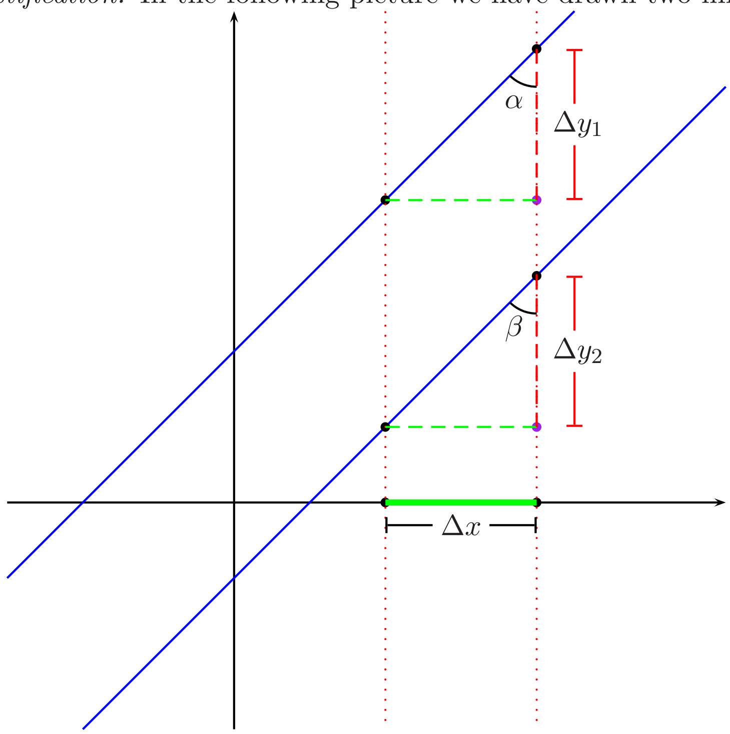
   (b) Graph all three lines in the same grid.



From our investigations so far it seems that if two lines have the same slope then they are *parallel*, that is they never cross each other no matter for how long we extend them. Indeed this is the case. To understand why we need some elementary geometry: some facts about similar triangles and some facts about parallel lines.

Fact 1. *If two* non-vertical *lines are parallel then they have the same slope. Conversely, if two lines have the same slope then they are parallel.*

Justification. In the following picture we have drawn two lines and with two points in each line.



If the two lines are parallel then $\alpha = \beta$ and the right triangles are congruent. It follows that $\Delta y_1 = \Delta y_2$ and therefore

$$\frac{\Delta y_1}{\Delta x} = \frac{\Delta y_2}{\Delta x}$$

So the slopes of the two lines are equal.

Conversely if the slopes of the two lines are equal then we have

$$\frac{\Delta y_1}{\Delta x} = \frac{\Delta y_2}{\Delta x}$$

and therefore

$$\Delta y_2 = \Delta y_1$$

The two right triangles are then equal and it follows that $\alpha = \beta$. So the two lines are parallel. □

Now lets see how this fact can be used.

1. A line passes through $(2, 1)$ and is parallel to the line with equation $y = 4x - 8$. Find an equation for this line.

2. A line is parallel to the line with equation $2x - 3y = -5$. Find the equation of this line if it cuts the y–axis at the point $(-3, 5)$.

3. A line passes through $(3, -5)$ and is parallel to the line with equation $x = -\frac{1}{2}$. Find the equation of the line.

Perpendicular lines

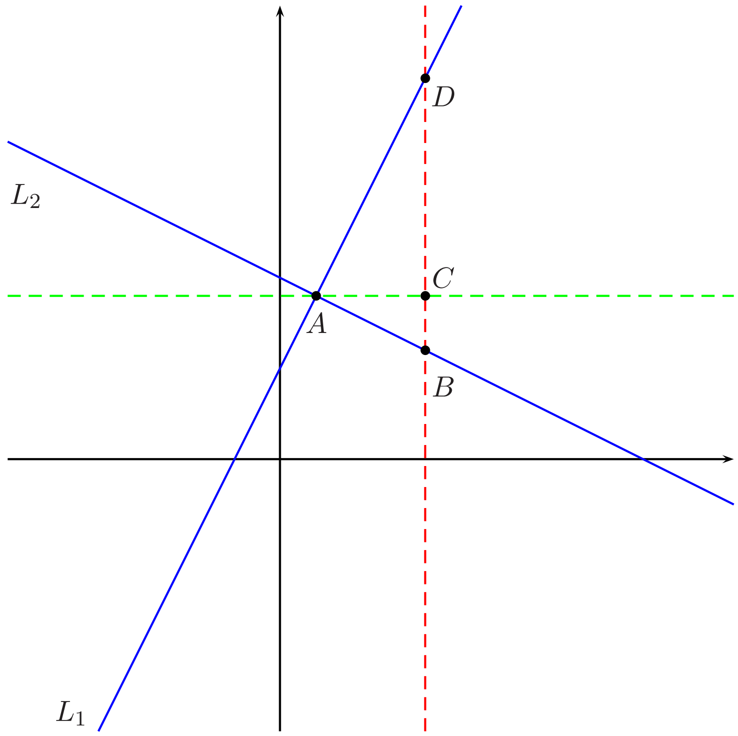
Two lines are called *perpendicular* or *orthogonal* if they intersect in right angles (i.e. $90°$).

Fact 2. *If two* non-vertical *lines are perpendicular then their slopes m_1 and m_2 satisfy the equation*

$$m_1 m_2 = -1$$

Conversely, if the slopes of two lines satisfy this equation then the two lines are perpendicular.

Justification. Consider the following figure, where the green line is parallel to the x–axis and the red line is parallel to the y–axis.



Notice the slope of L_1 is equal to $m_1 = \dfrac{|CD|}{|AC|}$ while the slope of L_2 is equal to[3] $m_2 = -\dfrac{|BC|}{|AC|}$. Now, if the two lines are perpendicular, the angle CAB is complementary[4] to the angle CAD. So the angles ABC and CAD are equal (being complementary to the same angle). It follows then that the right triangles ACD and BCA are similar. Therefore we have that

$$\frac{|CD|}{|AC|} = \frac{|AC|}{|BC|} \tag{7}$$

It follows then that

$$m_1 m_2 = -1 \tag{8}$$

Conversely, if equation (8) holds then equation (7) follows and we conclude that the right triangles ACD and BCA are similar. Therefore we have that the angles ABC and CAD are equal. So the angles CAB and CAD are complementary and therefore the lines L_1 and L_2 are perpendicular. □

Example 4. Verify that the lines with equations $2x+4y = 7$ and $4x-2y+5 = 0$ are perpendicular.

Answer. Solving the two equations for y we get $y = -\dfrac{x}{2} + \dfrac{7}{2}$ and $y = 2x - \dfrac{5}{2}$, respectively. So the first line has slope $-\dfrac{1}{2}$ while the second has slope 2. So the two lines are perpendicular. □

[3] Why the negative sign?

[4] Recall, that two angles are called complementary if their sum is 90°.

1. Find an equation for the line that contains the point $(-3, 2)$ and is perpendicular to the line with equation $y = \frac{3}{4}x - 7$.

2. Find the equation of the line that is perpependicular to the line with equation $x = 5$ and passes through the point $(-7, 6)$.

Preparation for next time

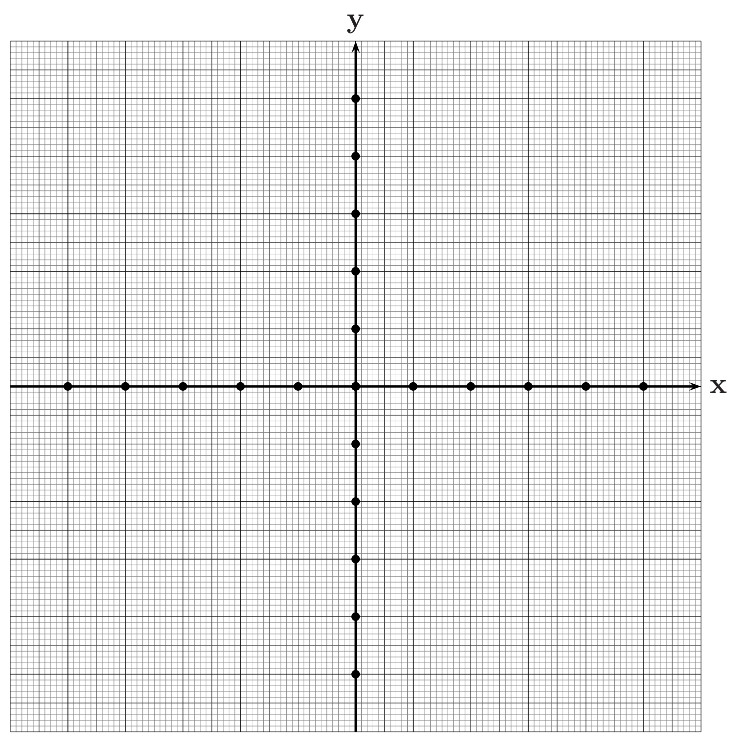
If we draw two lines in the plane we have exactly three possibilities:

- They intersect in exactly one point.
- They are parallel.
- They coincide (i.e. one is drawn on top of the other.)

In the next chapter we will see how to tell *algebraically*[5] in which case we are, and furthermore when we are in the first case how to find the coordinates of the point of intersection. For the moment lets explore this questions.

1. Consider the two lines $L_1 : 2x - 3y = 6$ and $L_2 : y = -1$.
   (a) Prove that this two lines will intersect in exactly one point *without* actually graphing them.

   (b) Graph the two lines in the same grid, and find their intersection point by inspection.



[5] *Algebraically* here means "using only the equations of the lines without the need to graph them".

(c) Verify that the point you found really belongs to both lines.

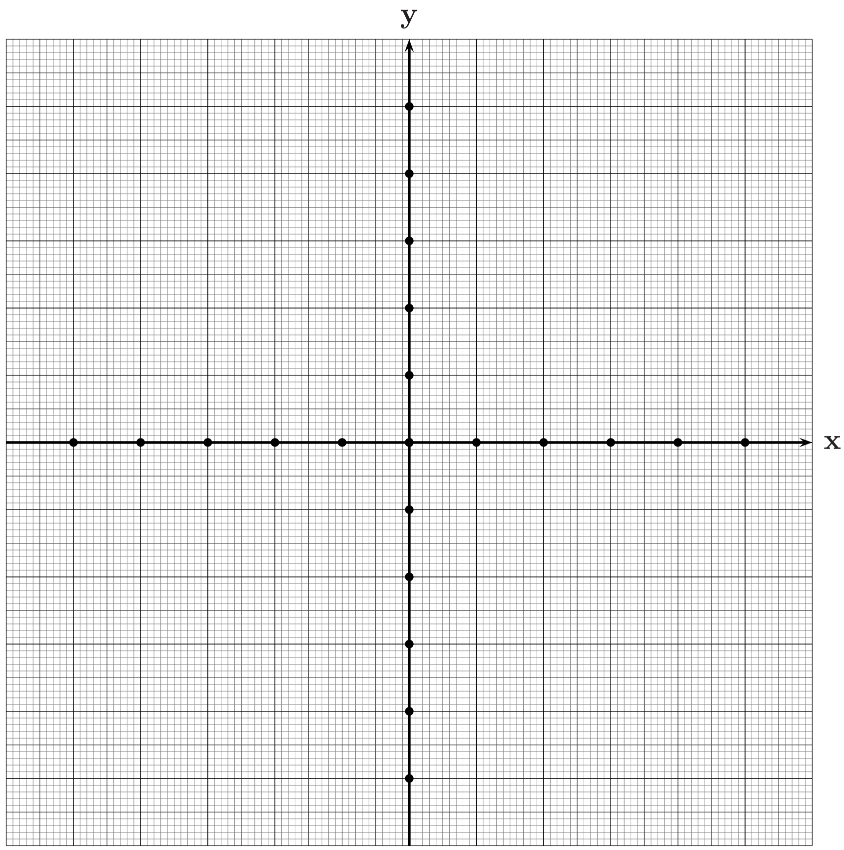
(d) Look at the equations again. How can we find a point that satisfies both equations algebraically?

2. Consider the two lines $L_1 : 2x + y = -2$ and $L_2 : x = 1$.

(a) Prove that these two lines will intersect at exactly one point, without graphing them.

(b) Still without graphing the lines, find their point of intersection.

(c) Graph the two lines in the same grid to verify your calculation.



3. Can you find the point of intersection of the lines $3x + 4y = 10$ and $y = x + 1$, algebraically?