Second Quiz for CSI35

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Directions: This quiz is due Thursday September 25, at 2:00 PM.

1. Recall that a *bit string* is a word on the alphabet $\{0, 1\}$. Let O be the function that counts the number of zeros in a bit string s.
   (a) Give a recursive definition of $O(s)$.
   (b) Use structural induction to prove that for two string bits s and t we have:
   $$O(st) = O(s) + O(t)$$
   where, st stands for the concatenation of the two strings s and t.

2. For a rooted tree T let $n(T)$ and $e(T)$ denote the number of vertices and the number of edges of T respectively. Use structural induction to prove that for all rooted trees T,
   $$n(T) - e(T) = 1$$

3. Let T be a full binary tree. Use structural induction to show that $n(T) \geq 2h(T) + 1$ where $n(T)$ is the number of vertices of T and $h(T)$ is the height of T.

4. Recall the definition of the Fibonacci numbers $\{f_n\}$:
   $$f_n = \begin{cases} 0 & \text{if n=0,} \\ 1 & \text{if n=1,} \\ f_{n-1} + f_{n-2} & \text{otherwise} \end{cases}$$
   Prove that for all positive integers n we have:
   $$f_{n+1}f_{n-1} - f_n^2 = (-1)^n$$

5. **Extra Credit:** Let $\varphi = \dfrac{1+\sqrt{5}}{2}$, $\bar{\varphi} = \dfrac{1-\sqrt{5}}{2}$ and let f_n be the nth Fibonacci number for $n \in \mathbb{N}$.
   (a) Prove that $\forall n \in \mathbb{N} \quad \varphi^n = f_{n-1} + f_n\varphi$ and $\bar{\varphi}^n = f_{n-1} + f_n\bar{\varphi}$.
   (b) Prove that
   $$\forall n \in \mathbb{N} \quad f_n = \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}}$$