Final exam for CSI35

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1. Let n be a positive integer. Prove that:

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right)=n+1$$

2. For an integer n let g_n be the number of ways that n can be writen as a sum of ones and twos, where the order that the summands are writen is important. For example, $g(1)=1$, $g(2)=2$ since 2 can be written either as 2 or as $1+1$, and $g(3)=3$ because 3 can be written as $1+1+1$ or as $1+2$ or as $2+1$.
   (a) Find a recursive definition of $g(n)$
   (b) Prove that this recursive definition is correct.

3. Let f_n denote the nth Fibbonacci number. Prove that
   (a) $\forall n \geq 1, \qquad f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$
   (b) $\forall n \geq 0, \qquad f_n^2 + f_{n+1}^2 = f_{2n+1}$

4. Let a_n be the sequence defined recursively by

$$a_n = \begin{cases} 1 & \text{if } n = 1, \\ \dfrac{n}{2}\displaystyle\sum_{k=1}^{n-1}\binom{n-2}{k-1}a_k a_{n-k} & \text{if } n > 1 \end{cases}$$

   (a) Find the first six terms of this sequence.
   (b) Conjecture a closed formula for a_n.
   (c) Test your conjecture by calculating further terms of the sequence.

5. Consider the poset $(\mathbb{Z}^+, |)$.
   (a) Find all upper bounds of the set $\{4, 6\}$. Is there a least upper bound?
   (b) Find all lower bounds of the set $\{8, 12\}$. Is there a greatest lower bound?

6. Can you find a graph with six vertices with degrees $1, 2, 3, 4, 5, 6$?

7. Refer to Figure 1. Are the graphs (a) and (b) isomorphic? How about (c) and (d)? Justify your answer, in particular if your answer is affirmative you should provide an isomorphism.

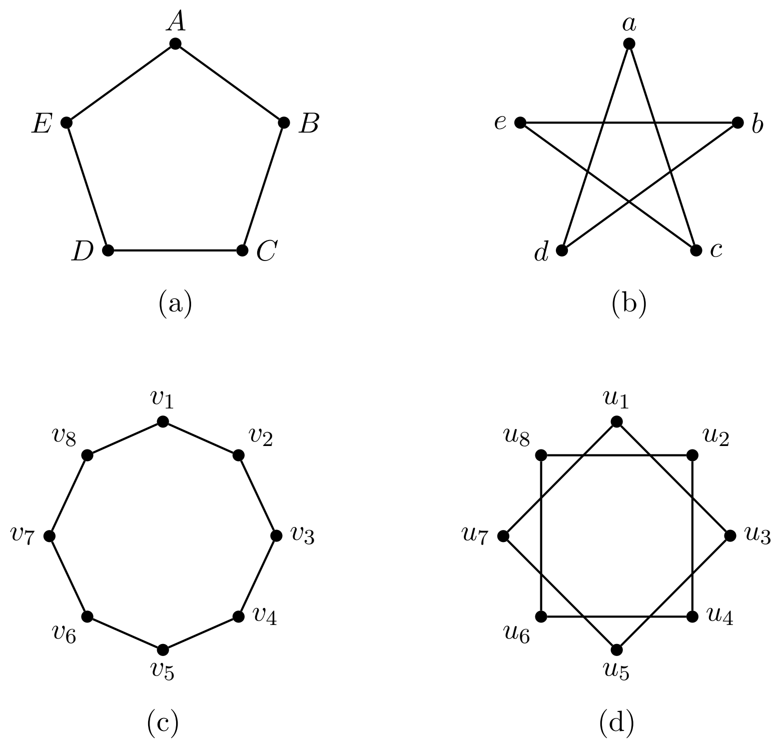


Figure 1: The graphs of question 7

8. Count von Diamond has been murdered in his estate. The internationally known detective (and part time graph theorist) Inspector Clouseau has been called in to investigate. The butler claims that he saw the gardener enter the pool room (where the murder took place) and then, shortly after, leave that room by the same door. On the other hand, the gardener says that he cannot be the man that the butler saw because he entered the house, went through each door exactly once and then left the house. Inspector Clouseau checks the floor plan (see Figure 2) and within minutes declares the case solved. Who done it?

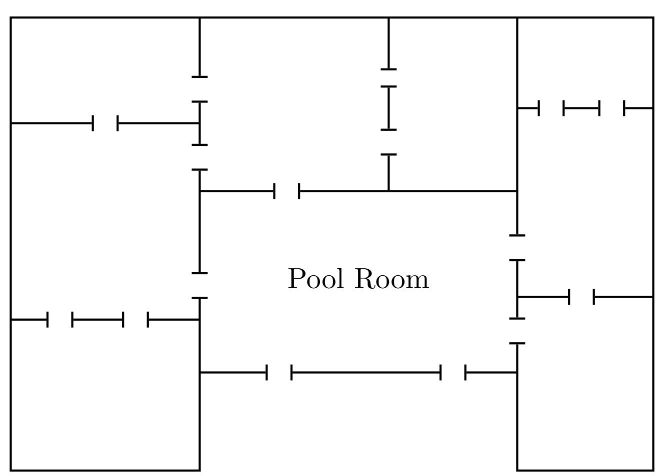
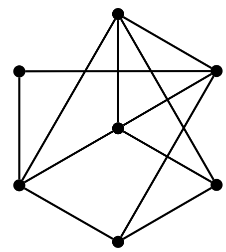
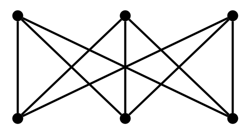


Figure 2: The floor of Von Diamond Estate

9. Does the following graph have an Euler circuit? Does it have an Euler path?



10. Does the following graph have an Hamilton circuit? Does it have an Hamilton path?



11. For which values of n is the complete graph K_n Eulerian?

12. List all possible trees with five or less vertices.

13. Let $\mathbb{M}_3$ be the set of 3×3 matrices and Δ_3 be the set of *upper triangular* 3×3 matrices:

$$\Delta_3 = \{(a_{ij}) \in \mathbb{M}_3 \mid a_{21} = a_{31} = a_{32} = 0\}$$

Consider the following relation on $\mathbb{M}_3$

$$R = \{(A, B) \in \mathbb{M}_3^2 \mid A - B \in \Delta_3\}$$

   (a) Prove that R is an equivalence relation.

   (b) Find the equivalence class of

$$A = \begin{pmatrix} 1 & -4 & 5 \\ -3 & 9 & 7 \\ 5 & 7 & 3 \end{pmatrix}$$

14. Consider the Petersen graph as shown Figure 3.

   (a) Find a Hamiltonian path in the Petersen graph.

   (b) Prove that the Petersen graph does not have a Hamilton circuit.
   Hint. One way to proceed is the following: Prove first that if there is a Hamilton circuit, say c, then in c there must be an even number of edges connecting the outer pentagon to the inner star. Then proceed case by case: prove that there is no c with two edges connecting the pentagon to the star and then that there is no c that has four such edges either.

15. Draw the game tree for the game of nim if the starting position consists of three piles with one, two and three stones respectively. Which player has a winning strategy?

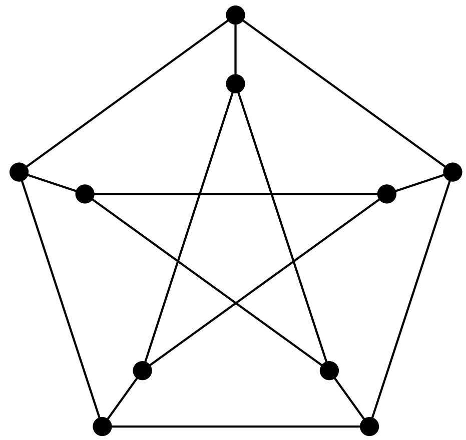


Figure 3: The Petersen graph

16. How many children does the root of the game tree for nim have if the starting position consists of three piles with seven, five and three stones respectively?
17. You should be able to anwer one (or both) of the following questions:
    (a) Give a solution to the Eight-Queens puzzle.
    (b) Show a Knight's Tour.