Review for the midterm for Math 31

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1. Calculate the following limits. If a limit does not exist state so and explain why.

   (a) $\lim_{x\to 0} \cos(x + \sin x)$

   (b) $\lim_{x\to 0} \frac{x^2 - 9}{x^2 + 2x - 3}$

   (c) $\lim_{x\to 1^+} \frac{x^2 + x - 6}{x^2 + 2x - 3}$

   (d) $\lim_{x\to -4} \frac{x + 4}{|x + 4|}$

   (e) $\lim_{z\to 3} \frac{z + 2}{z^2 - z - 6}$

   (f) $\lim_{w\to -2} \frac{w + 1}{4 + 8w + 5w^2 + w^3}$

   (g) $\lim_{t\to 2} \frac{2t^2 + 7t + 6}{t^4 - 16}$

   (h) $\lim_{z\to 2^-} \frac{|z - 2|}{z - 2}$

   (i) $\lim_{x\to 9} \frac{3 - \sqrt{x}}{x - 9}$

   (j) $\lim_{u\to 2} \frac{\sqrt{x + 2} - \sqrt{2x}}{x^2 - 2x}$

   (k) $\lim_{t\to 25} \frac{\sqrt{t} - 5}{\sqrt{t - 25}}$

   (l) $\lim_{x\to \pi/2^+} \tan x$

   (m) $\lim_{x\to 0} sin\frac{1}{x}$

   (n) $\lim_{x\to 0} x^4 \sin\frac{1}{x}$

   (o) $\lim_{x\to 0} \frac{\sin 5x}{x}$

   (p) $\lim_{x\to -3} \frac{\sin(x + 3)}{x^2 + x - 6}$

   (q) $\lim_{\phi\to 0} \frac{\sin(cos(\phi))}{\sec\phi}$

2. Find the real number(s) a so that f is continuous at all real numbers:

$$f(x) = \begin{cases} x^2 - 2a & \text{if } x < -1 \\ 3x + 3a & \text{if } x \geq -1 \end{cases}$$

3. A Tibetan monk leaves the monastery at 7:00 A.M. and takes his usual path to the top of the mountain, arriving at 7:00 P.M. The following morning, he start at 7:00 A.M. at the top of the mountain and takes the same path back, arriving at the monastery at 7:00 P.M. Prove that there is at least one point of the path that the monk will cross at exactly the same time both days.

4. Prove that the equation $2^x = x^2$ has a solution in the interval $[-1, 0]$.

5. Give an example of a function that
   (a) has a jump discontinuity at $x = -5$.
   (b) has a removable singularity at $x = 0$.
   (c) has an infinite discontinuity at $x = 3$.
   (d) is continuous everywhere except at $x = 0$ and the discontinuity is not jump, removable or infinite.

6. Calculate each of the following derivatives using the definition of the derivative as the limit of the difference quotients:
   (a) $\frac{d}{dx}\left(x^3 - 3x^2 + 5x - 2\right)$
   (b) $\left(\frac{1}{x^2}\right)'$
   (c) $\left(\frac{x+1}{x+2}\right)'$
   (d) $\frac{d}{dx}\left(\sqrt{2-3x}\,\right)$

7. Find an equation of the tangent to the curve at the given point:
   (a) $y = 4\sin^2 x$, at the point $(\frac{\pi}{6}, 1)$
   (b) $y = \frac{x^2 - 4}{x^2 + 4}$, at the point $(0, -1)$
   (c) $y = \sqrt{4 - 2\sin x}$, at the point $(0, 2)$
   (d) $x^3 + 3x^2 y - 2xy^2 - y^3 = 15$, at the point $(2, 3)$
   (e) $x^{2/3} + y^{2/3} = 4$, at the point $(-3\sqrt{3}, 1)$

8. A particle moves on a vertical line according to the law of motion

$$s(t) = t^3 - 27t + 4, \qquad t \geq 0$$

   where t is measured in seconds and s in meters.
   (a) Find the velocity and the acceleration of the particle at time t.
   (b) When is the particle moving upward and when is it moving downward?

(c) Find the total distance traveled by the particle during the first six seconds.

9. Boat A travels west at a 50 miles per hour and boat B travels north at 60 miles per hour. The two boats are going to collide in 3 hours. At what rate are the two boats approaching each other 1 hour before the collision?

10. The graph of f', the derivative of a function f is shown in Figure 1. Draw a possible graph for f.

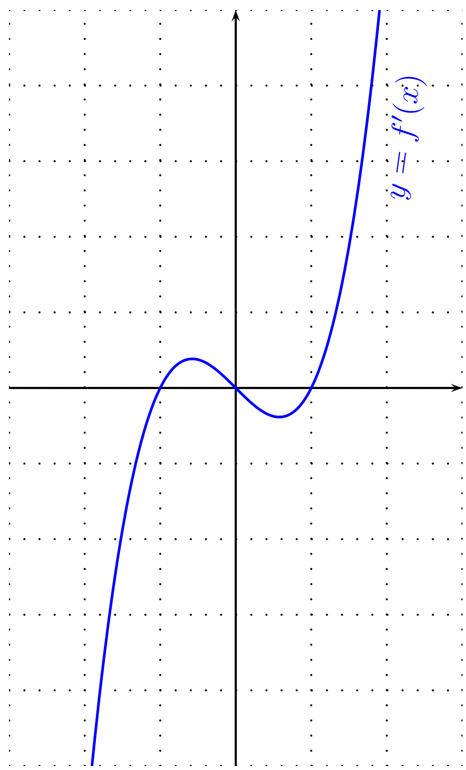

Figure 1: The graph of the derivative of a function

11. For each of the following functions find the intervals of increase or decrease, local maximum or minimum values, the intervals of concavity, and inflection points. Then use this information to sketch a graph of the function.

(a) $f(x) = 2 - 2x - x^3$

(b) $f(x) = x^3 - 6x^2 - 15x + 4$

(c) $f(x) = x^4 - 3x^3 + 3x^2 - x$

(d) $f(x) = \dfrac{1}{1 - x^2}$

(e) $f(x) = x\sqrt{x + 4}$

(f) $g(x) = \sin^2 x - 2\cos x$

(g) $h(x) = \sqrt{x} - \sqrt[3]{x}$

(h) $g(x) = x - \sqrt{1-x}$

(i) $f(x) = \dfrac{x^2}{\sqrt{x+1}}$

(j) $g(x) = x + \sqrt{|x|}$

(k) $h(x) = |x^2 - x - 2|$

12. Give an example of a function f that has a tangent line at $x = 1$ but $f'(1)$ does not exist.

13. Let $f(x) = \sin x^2$ defined on the interval $[-3, 3]$. Show that the premises of Rolle's theorem are satisfied. Then find all numbers c that satisfy the conclusion of the theorem.

14. Prove that that the equation $\cos x = x$ has exactly one solution in the interval $(0, 1)$.

15. Prove that the following function has exactly one real root:

$$3x^5 + x^3 + 6x - 5 = 0$$

16. Two stationary patrol cars equipped with radar are 20 miles apart on a highway. As a truck passes the first patrol car, its speed is checked to 55 miles per hour. Fifteen minutes later, when the truck passes the second patrol car its speed is checked to 52 miles per hour. If the speed limit is 55 miles per hour explain why the truck could still get a speeding ticket.

17. Can you give an example of a *non-constant* function f such that $f'(x) = 0$ for all x in the domain of f?