Review for the midterm for Math 31
The answers
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1. Calculate the following limits. If a limit does not exist state so and explain why.

(a) $\lim_{x\to 0} \cos(x+\sin x) = 1$

(b) $\lim_{x\to 0} \frac{x^2-9}{x^2+2x-3} = 3$

(c) $\lim_{x\to 1^+} \frac{x^2+x-6}{x^2+2x-3} = +\infty$

(d) $\lim_{x\to -4} \frac{x+4}{|x+4|}$ does not exist

(e) $\lim_{z\to -2} \frac{z+2}{z^2-z-6}$ does not exist

(f) $\lim_{w\to -2} \frac{w+1}{4+8w+5w^2+w^3} = +\infty$

(g) $\lim_{t\to -2} \frac{2t^2+7t+6}{t^4-16} = \frac{1}{32}$

(h) $\lim_{z\to 2^-} \frac{|z-2|}{z-2} = -1$

(i) $\lim_{x\to 9} \frac{3-\sqrt{x}}{x-9} = \frac{1}{6}$

(j) $\lim_{x\to 2} \frac{\sqrt{x+2}-\sqrt{2x}}{x^2-2x} = -\frac{1}{8}$

(k) $\lim_{t\to 25} \frac{\sqrt{t}-5}{\sqrt{t-25}} = 0$

(l) $\lim_{x\to \pi/2^+} \tan x = -\infty$

(m) $\lim_{x\to 0} \sin\frac{1}{x}$ Does not exist

(n) $\lim_{x\to 0} x^4 \sin\frac{1}{x} = 0$

(o) $\lim_{x\to 0} \frac{\sin 5x}{x} = \frac{1}{5}$

(p) $\lim_{x\to -3} \frac{\sin(x+3)}{x^2+x-6} = -\frac{1}{5}$

(q) $\lim_{\phi\to 0} \frac{\sin(cos(\phi))}{\sec\phi} = \sin 1$

2. Find the real number(s) a so that f is continuous at all real numbers:

$$f(x) = \begin{cases} x^2 - 2a & \text{if } x < -1 \\ 3x + 3a & \text{if } x \geq -1 \end{cases}$$

Answer. $a = \dfrac{4}{5}$ □

3. A Tibetan monk leaves the monastery at 7:00 A.M. and takes his usual path to the top of the mountain, arriving at 7:00 P.M. The following morning, he start at 7:00 A.M. at the top of the mountain and takes the same path back, arriving at the monastery at 7:00 P.M. Prove that there is at least one point of the path that the monk will cross at exactly the same time both days.

   Answer. Let M be the distance from the monastery to the top of the mountain. For convenience use time measurements so that $t = 0$ corresponds to 7:00 AM and $t = 1$ corresponds to 7:00 PM. Let $s = s(t)$ be the position of the monk during the first day and $pp(t)$ his position during the second day. Consider the function $f(t) = s(t) - p(t)$. Then we have:

$$f(0) = s(0) - p(0) = 0 - M = M$$
$$f(1) = s(1) - p(1) = M - 0 = -M$$

   It follows by the Intermidiate Value Theorem that for some time t_0 we must have $f(t_0) = 0$, that is $s(t_o) = p(t_0)$. In other words the monk is at the same position exactly the same time both days. □

4. Prove that the equation $2^x = x^2$ has a solution in the interval $[-1, 0]$.

   Answer. Let $f(x) = 2^x - x^2$. Then f is continuous and we have that $f(-1) = -\dfrac{1}{2}$ while $f(0) = 1$. Since 0 is between these two values it follows from the I.V.T. that for at least one point c in the interval $[-1, 0]$ we must have $f(c) = 0$. In other words, there is a $c \in [-1, 0]$ that solves the equation $2^x = x^2$. □

5. Give an example of a function that
   (a) has a jump discontinuity at $x = -5$.
   (b) has a removable singularity at $x = 0$.
   (c) has an infinite discontinuity at $x = 3$.
   (d) is continuous everywhere except at $x = 0$ and the discontinuity is not jump, removable or infinite.

   Proof. These are some possible answers:

   (a) $f(x) = \begin{cases} x & \text{if } x < -5 \\ x^2 & \text{if } x \geq -5 \end{cases}$

(b) $g(x) = \begin{cases} \sin x & \text{if } x \neq 0 \\ 7 & \text{if } x = 0 \end{cases}$

(c) $h(x) = \dfrac{2}{x-3}$

(d) $f(x) = \sin \dfrac{1}{x}$

$\square$

6. Calculate each of the following derivatives using the definition of the derivative as the limit of the difference quotients:

(a) $\dfrac{d}{dx}\left(x^3 - 3x^2 + 5x - 2\right)$

(b) $\left(\dfrac{1}{x^2}\right)'$

(c) $\left(\dfrac{x+1}{x+2}\right)'$

(d) $\dfrac{d}{dx}\left(\sqrt{2-3x}\,\right)$

Answer. These were done in class. $\square$

7. Find an equation of the tangent to the curve at the given point:

(a) $y = 4\sin^2 x$, at the point $(\frac{\pi}{6}, 1)$ $\quad y = 2\sqrt{3}x + 1 - \dfrac{\pi\sqrt{3}}{3}$

(b) $y = \dfrac{x^2-4}{x^2+4}$, at the point $(0, -1)$ $\quad y = -1$

(c) $y = \sqrt{4 - 2\sin x}$, at the point $(0, 2)$ $\quad y = 2 - \dfrac{x}{2}$

(d) $x^3 + 3x^2y - 2xy^2 - y^3 = 15$, at the point $(2, 3)$ $\quad y = \dfrac{10}{13}x + \dfrac{19}{13}$

(e) $x^{2/3} + y^{2/3} = 4$, at the point $(-3\sqrt{3}, 1)$ $\quad y = \dfrac{\sqrt{3}}{3}x + 4$

8. A particle moves on a vertical line according to the law of motion

$$s(t) = t^3 - 27t + 4, \qquad t \geq 0$$

where t is measured in seconds and s in meters.

(a) Find the velocity and the acceleration of the particle at time t.

(b) When is the particle moving upward and when is it moving downward?

(c) Find the total distance traveled by the particle during the first six seconds.

Answer. (a) Velocity is $s'(t) = 3t^2 - 27$ and acceleration is $s''(t) = 6t$.

(b) The particle is moving downwards for $0 \leq t \leq 3$ and upwards for $t \geq 3$.

(c) During the first 3 time units the particle traveled downwards from $s = 4$ to $s = -50$ so it covered a distnace of 54 units. During the last 3 time units the particle traveled upwards from $s = -50$ to $s = 58$ so it covered a distance of 108 units. So the total distance traveled by the particle is $108 + 54 = 162$ units.

□

9. Boat A travels west at a 50 miles per hour and boat B travels north at 60 miles per hour. The two boats are going to collide in 3 hours. At what rate are the two boats approaching each other 1 hour before the collision?

Answer. □

10. The graph of f', the derivative of a function f is shown in Figure 1. Draw a possible graph for f.

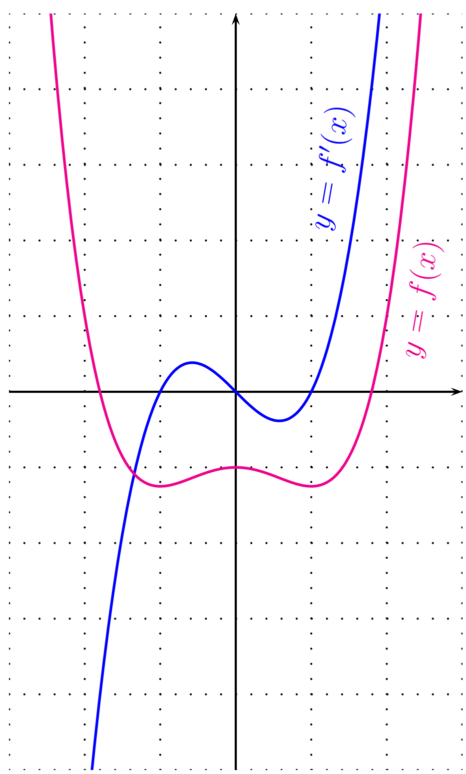

Figure 1: The graph of the derivative of a function

Answer. A possible graph is shown. □

11. For each of the following functions find the intervals of increase or decrease, local maximum or minimum values, the intervals of concavity, and inflection points. Then use this information to sketch a graph of the function.

   (a) $f(x) = 2 - 2x - x^3$,
   (b) $f(x) = x^3 - 6x^2 - 15x + 4$
   (c) $f(x) = x^4 - 3x^3 + 3x^2 - x$
   (d) $f(x) = \dfrac{1}{1-x^2}$
   (e) $f(x) = x\sqrt{x+4}$
   (f) $g(x) = \sin^2 x - 2\cos x$
   (g) $h(x) = \sqrt{x} - \sqrt[3]{x}$
   (h) $g(x) = x - \sqrt{1-x}$
   (i) $f(x) = \dfrac{x^2}{\sqrt{x+1}}$
   (j) $g(x) = x + \sqrt{|x|}$
   (k) $h(x) = |x^2 - x - 2|$

Answer. (a) There are no critical points and the first derivative is always negative. So f is decreasing on $\mathbb{R}$. The second derivative vanishes at $x = 0$ and its positive on $(-\infty, 0)$ and negative at $(0, \infty)$. Thus the function is concave upward $(-\infty, 0)$, concave downward in $(0, \infty)$ and $x = 0$ is an inflection point.

(b) There are two critical points $x = -1$ and $x = 5$. The first derivative is positive on $(-\infty, -1)$ and $(5, \infty)$ and negative on $(-1, 5)$. So f is increasing on $(-\infty, -1)$ and $(5, \infty)$ and decreasing on $(-1, 5)$. It follows that we have a local maximum at $x = -1$ with maximum value $f(-1) = 12$, and a local minimum at $x = 5$ with minimal value $f(5) = -96$. The second derivative is negative on $(-\infty, 2)$, vanishes at $x = 2$ and positive at $(2, \infty)$. So f is concave downwards in $(-\infty, 0)$ and concave upwards in $(2, \infty)$. At $x = 2$ we have an inflection point.

(c) The first derivative vanishes at $x = \dfrac{1}{4}$ and at $x = 1$ which is a double root. Also $f'(x)$ is negative for $x \in (-\infty, \dfrac{1}{4})$ and positive for $x \in (\dfrac{1}{4}, 1)$ and for $x \in (1, \infty)$. So f is decreasing for $x \in (\dfrac{1}{4}, 1)$ and increasing for $x \in (\dfrac{1}{4}, \infty)$. It follows that $x = \dfrac{1}{4}$ is a local minimum with minimal value $f(\dfrac{1}{4}) = -\dfrac{27}{256}$. The second derivative vanishes at $x = \dfrac{1}{2}$ and $x = 1$, it is positive on $(-\infty, \frac{1}{2})$ and $(1, \infty)$ and negative on $(\dfrac{1}{2}, 1)$. So the function is concave upwards on $(-\infty, \dfrac{1}{2})$ and on $(1, \infty)$ and it is concave upwards on $(1, \infty)$. Also we have inflection points at $x = \dfrac{1}{2}$ and $x = 1$.

(d) We worked this one in class. The domain of the function is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$. The first derivative vanishes at $x=0$, is negative on the intervals $(-\infty,-1)$, $(-1,0)$ and is positive on the intervals $(0,1)$ and $(1,\infty)$. We have a minimum at $x=0$ with minimum value $f(0)=1$. The second derivative has no roots. It is negative on $(-\infty,-1)$ and $(1,\infty)$ and positive on $(-1,1)$. So the fucnction is concave downwards on $(-\infty,-1)$ and $(1,\infty)$ and concave upwards on $(-1,1)$.

$\square$

12. Give an example of a function f that has a tangent line at $x=1$ but $f'(1)$ does not exist.

Answer. $f(x)=(x-1)^{2/3}$. At $x=1$ we have a vertical tangent line. $\square$

13. Let $f(x)=\sin x^2$ defined on the interval $[-3,3]$. Show that the premises of Rolle's theorem are satisfied. Then find all numbers c that satisfy the conclusion of the theorem.

Answer. (a) The function is continuous on $[-3,3]$

(b) The function is differentiable on $(-3,3)$.

(c) $f(3)=\sin 9$ and $f(-3)=\sin 9$. Thus $f(-3)=f(3)$

So the premises of Rolle's theorem are satisfied. The conclusion of Rolle's theorem then states that there is a c in $(-3,3)$ such that $f'(c)=0$. This is equivalent to

$$2c\cos c^2=0 \Longleftrightarrow c=0 \quad \text{or} \quad \cos c^2=0$$

Now since c^2 is never negative we have

$$\begin{aligned}\cos c^2=0 &\Longleftrightarrow c^2=\frac{\pi}{2} \quad \text{or} \quad c^2=\frac{3\pi}{2}\\ &\Longleftrightarrow c=\pm\sqrt{\frac{\pi}{2}} \quad \text{or} \quad c=\pm\sqrt{\frac{3\pi}{2}}\end{aligned}$$

Thus there are five c that verify the conclusion of Rolle's theorem:

$$c=-\sqrt{\frac{3\pi}{2}},\, c=-\sqrt{\frac{\pi}{2}},\, c=0,\, c=\sqrt{\frac{\pi}{2}},\, c=\sqrt{\frac{3\pi}{2}}$$

$\square$

14. Prove that that the equation $\cos x=x$ has exactly one solution in the interval $(0,1)$.

Answer. Consider the function $f(x)=x-\cos x$. f is continuous on $[0,1]$ and $f(0)=-1$ while $f(1)=1-\cos 1>0$. So by I.V.T. there is a point c in $(0,1)$ such that $f(c)=0$, in other words $c=\cos c$. Thus the given equation has at least one solution in $(0,1)$.

Now if there were two solutions in that interval then by Rolle's theorem $f'(x)$ would have a root in $(0,1)$. However

$$f'(x)=1+\sin x$$

and for $x \in (0, 1)$, $f'(x) > 1$. Therefore $f(x)$ cannot have more than one roots in $(0, 1)$. □

15. Prove that the following function has exactly one real root:

$$3x^5 + x^3 + 6x - 5 = 0$$

Answer. Let $f(x) = 3x^5 + x^3 + 6x - 5$ Looking at the interval $[0, 1]$ we see that $f(0) = -5$ and $f(1) = 3$. So since 0 is between these two values $f(x)$ has at least one root.

By Rolle's theorem if f has two roots then f' would have at least one root between those two roots. However,

$$f'(x) = 15x^4 + 3x^2 + 6$$

is always positive. Therefore f cannot have two real roots. □

16. Two stationary patrol cars equipped with radar are 20 miles apart on a highway. As a truck passes the first patrol car, its speed is checked to 55 miles per hour. Fifteen minutes later, when the truck passes the second patrol car its speed is checked to 52 miles per hour. If the speed limit is 55 miles per hour explain why the truck could still get a speeding ticket.

Answer. Since the two parols are 20 miles apart and the truck took 15 minutes to cover the distance between them it follows that its average speed was 80 miles per hour. By the Mean Value Theorem at some time between the two patrols the instanteneous speed was exactly 80 miles per hour. So even though its speed was bellow the speed limit at the check points we know that it went over the limit at some point between the check points. □

17. Can you give an example of a *non-constant* function f such that $f'(x) = 0$ for all x in the domain of f?

Answer. Let f be the function with domain $(-\infty, 0) \cup (0, \infty)$ given by:

$$f(x) = \begin{cases} 3 & \text{if } x < 0 \\ 42 & \text{if } x > 0 \end{cases}$$

□