The answers to the Midterm for Math31

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1. Calculate the following limits. If a limit does not exist state so and explain why.

   (a) $\lim_{x\to -4} \frac{x^2-16}{x^2+9x+20} = \lim_{t\to -4} \frac{x-4}{x+5} = -8$

   (b) $\lim_{x\to 0} \frac{\sin 7x}{x} = \lim_{x\to 0} 7\frac{\sin 7x}{7x} = 7\lim_{x\to 0} \frac{\sin 7x}{7x} = 7$

   (c) $\lim_{x\to 0} x^2 \cos\frac{1}{x}$

   Answer. We know that for all $x \neq 0$ we have

   $$-1 \leq \cos\frac{1}{x} \leq 1$$

   and therefore by multiplying with x^2 (this is permitted since $x^2 > 0$) we have:

   $$-x^2 \leq x^2\cos\frac{1}{x} \leq x^2$$

   Now $\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$ and therefore by the squeeze theorem we conclude that

   $$\lim_{x\to 0} x^2\cos\frac{1}{x} = 0$$

   □

   (d) $\lim_{x\to 2} \frac{|x-2|}{x-2}$

   Answer. We have that

   $$\frac{|x-2|}{x-2} = \begin{cases} -1 & \text{if } x < 2 \\ 1 & \text{if } x > 2 \end{cases}$$

   Therefore,

   $$\lim_{x\to 2^-} \frac{|x-2|}{x-2} = -1$$

   while

   $$\lim_{x\to 2^+} \frac{|x-2|}{x-2} = 1$$

   Since the two one-sided limits disagree it follows that the limit does not exist. □

   (e) $\lim_{z\to -2^-} \frac{z+2}{z^2-z-6} = \lim_{z\to -2^-} \frac{1}{z-3} = -\frac{1}{5}$

2. Find all real numbers a so that f is continuous at all real numbers, where

   $$f(x) = \begin{cases} x^2+2a & \text{if } x \leq 5 \\ 2x+a^2 & \text{if } x > 5 \end{cases}$$

Answer. The function f is equal to a polynomial for $x < 5$ and for $x > 5$ and therefore it is continuous on $(-\infty, 5) \cup (5, \infty)$. So in order for f to be continuous at all real numbers we need f to be continuous at $x = 5$. In other words we need

$$\lim_{x \to 5} f(x) = f(5)$$

Now,

$$\begin{aligned} \lim_{x \to 5^-} f(x) &= \lim_{x \to 5^-} x^2 + 2a \\ &= 25 + 2a \end{aligned}$$

while

$$\begin{aligned} \lim_{x \to 5^+} f(x) &= \lim_{x \to 5^+} 2x + a^2 \\ &= 10 + a^2 \end{aligned}$$

So for the limit to exist we need

$$25 + 2a = 10 + a^2 \iff a = -3 \text{ or } a = 5$$

In both cases the limit will equal the value of the function at $x = 5$ so f will be continuous at all real numbers when $a = -3$ or when $a = 5$. □

3. (a) Prove that the equation $x^3 + 3x^2 + 6x + 8 = 0$ has a solution in the interval $[-3, 10]$.

   Answer. Let $f(x) = x^3 + 3x^2 + 6x + 8$. We have $f(-3) = -10$ and $f(10) = 1368$. Now since $-10 < 0 < 1368$ it follows by the Intermediate Value Theorem that there is a $cin(-3, 10)$ so that $f(c) = 0$, that is, c is a solution of the given equation. □

   (b) Prove that the equation in part (a) has *only* one solution.

   Answer. A consequence of Rolle's theorem is that between any two zeroes of a differential function f there is a zero of its derivative f'; in particular, if f has two zeroes then f' has at least one zero. However, in our case we have

   $$f'(x) = 3x^2 + 6x + 6$$

   which is a quadratic function with discriminant $b^2 - 4ac = 36 - 72 < 0$, so that f' has no real zeros. It follows that f cannot have more than one real zeros. Since in part (a) we proved that f has at least one solution we conclude that f has exactly one solution. □

4. Calculate the following derivative using the definition of the derivative as the limit of the difference quotients

$$\left(\sqrt{x}\right)'$$

Answer. Let $f(x) = \sqrt{x}$. Then

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \frac{\sqrt{x+h}-\sqrt{x}}{h}\,\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \frac{1}{\sqrt{x+h}+\sqrt{x}}
\end{aligned}$$

So that,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{1}{\sqrt{x}+\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

□

5. Find an equation of the line tangent to the graph of

$$y = \frac{x^2+1}{x^2-1}$$

at the point with $x = 0$.

Answer. When $x = 0$ we have $y = -1$, so that the line is tangent at the point $(0, -1)$. The slope of the tangent line equals $y'(0)$. We calculate:

$$\begin{aligned}
y' &= \frac{2x(x^2-1)-(x^2+1)2x}{(x^2-1)^2} \\
&= \frac{2x^3-2x-2x^3-2x}{(x^2-1)^2} \\
&= \frac{-4x}{(x^2-1)^2}
\end{aligned}$$

Thus the slope of the tangent line is $y'(0) = 0$. Therefore the equation of the tangent line is:

$$y = -1$$

□

6. Find an equation of the line tangent to the graph of the equation

$$xy^2 - 3x^2y + x^3 + 1 = 0$$

at the point $(1, 2)$.

Answer. Assuming that $y = y(x)$ we can differentiate the equation:

$$\begin{aligned} xy^2 - 3x^2y + x^3 + 1 = 0 &\Longrightarrow \left(xy^2 - 3x^2y + x^3 + 1\right)' = 0 \\ &\Longrightarrow y^2 + 2xyy' - 6xy - 3x^2y' + 3x^2 = 0 \\ &\Longrightarrow y' = \frac{6yx - y^2 - 3x^2}{2yx - 3x^2} \end{aligned}$$

Substituting we $x = 1$ and $y = 2$ we obtain

$$y' = 5$$

So the equation of the tangent line is

$$y - 2 = 5(x - 1)$$

Or equivalently,

$$y = 5x - 3$$

□

7. A particle moves on a vertical line according to the law of motion

$$s(t) = t^3 + 3t^2 - 9t, \qquad t \geq 0$$

where t is measured in seconds and s in meters.

(a) Find the velocity and the acceleration of the particle at time t.

Answer. The velocity is $s'(t) = 3t^2 + 6t - 9$ and the acceleration is $s''(t) = 6t + 6$ □

(b) When is the particle moving upward and when is it moving downward?

Answer. The particle moves upwards when $s'(t) > 0$ and downwards when $s'(t) < 0$. Now

$$s'(t) = 3(t + 3)(t - 1)$$

so that $s'(t) < 0$ for $0 \leq t < 1$ and $s'(t) > 0$ for $t > 1$. So the particle moves downward for $t \in (0, 1)$ and downward for $t \in (1, \infty)$. □

(c) Find the total distance traveled by the particle during the first three seconds.

Answer. From the previous part we know that the particle starts at $t = 0$ and moves downward until $t = 1$ and then it turns and moves upward. During the downward movement it goes from $s(0) = 0$ to $s(1) = -5$, so it covers a distance of 5 meters. During the upward movement it goes from $s(1) = -5$ to $s(3) = 27$ so it covers a distance of $27 - (-5) = 32$ meters. So in total the particle covered a distance of $32 + 5 = 37$ meters. □

8. Let $f(x) = x^3 + 5x^2 + 6x - x^2 - 5x - 1$ defined on the interval $[-3, 1]$. Show that the premises of Rolle's theorem are satisfied. Then find all numbers c that satisfy the conclusion of the theorem.

   Answer. We have

   1. f is continuous on $[-3, 1]$ since it is a polynomial.
   2. f is differentiable on $(-3, 1)$ since it is a polynomial.
   3. $f(-3) = 5$ and $f(1) = 5$ so that $f(-3) = f(1)$

   so that the premises of Rolle's theorem are satisfied. The conclusion of the theorem is then that there is a c in $(-3, 1)$ such that $f'(c) = 0$. We have

   $$f'(x) = 3x^2 + 8x + 1$$

   so that $c \in (-3, 1)$ satisfies the premises of Rolle's theorem when

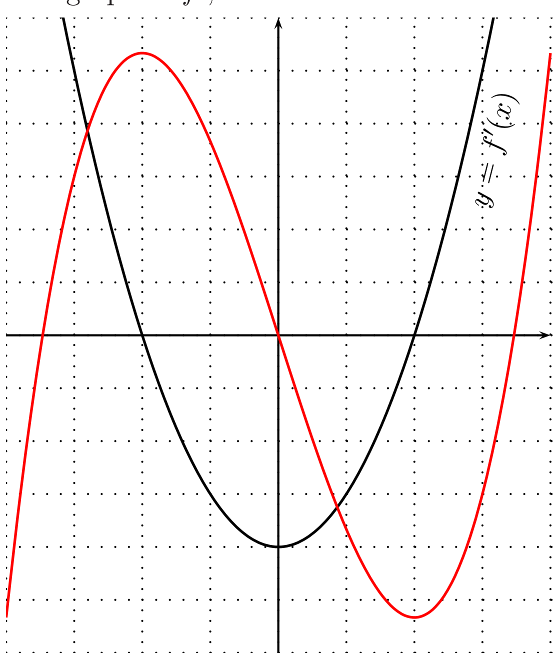
   $$3c^2 + 8c + 1 = 0$$

   The above equation has two solutions:

   $$c = \frac{-4 \pm \sqrt{13}}{3}$$

   Both of these solutions are in the interval $(-3, 1)$ and thus satisfy the conclusion of Rolle's theorem. $\square$

9. The graph of f', the derivative of a function f is shown bellow. Draw a possible graph for f.



	$-\infty$		-2		0		2		∞
$f'(x)$		$+$	0	$-$		$-$	0	$+$	
$f''(x)$		$-$		$-$	0	$+$		$+$	

10. Consider the following function:

    $$f(x) = 2x^3 - 9x^2 - 24x$$

    find the intervals of increase or decrease, local maximum or minimum values, the intervals of concavity, and inflection points.

Answer. We have

$$f'(x) = 6x^2 - 18x - 24$$
$$f''(x) = 12x - 18$$

The critical points of f are given by $6x^2 - 18x - 24 = 0 \iff x = 4$ or $x = -1$ For the sign of the first derivative we have the following table

	$-\infty$		-1		4		∞
$f'(x)$		$+$	0	$-$	0	$+$	

From the table we conclude that f is increasing at the intervals $(-\infty, -1)$, $(4, \infty)$ and decreasing in $(-1, 4)$. Therefore at $x = -1$ we have a local maximum value $f(-1) = 13$ and at $x = 4$ we have a local minimum value $f(4) = -112$.

For the second derivative we have that $f''(x) = 0 \iff x = \dfrac{3}{2}$ and that $f''(x) < 0$ for $x \in (-\infty, -\dfrac{3}{2})$ while $f''(x) > 0$ for $x \in (\dfrac{3}{2}, \infty)$. So f is concave downward for $x \in (-\infty, -\dfrac{3}{2})$ concave upward for $x \in (\dfrac{3}{2}, \infty)$ and at $x = \dfrac{3}{2}$ we have an inflection point. $\square$